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I.—ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS. No. II.

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IN the last number of the Journal, a method was given for the investigation of a class of differential equations, by means of successive reductions.

The present communication contains solutions of some analogous equations effected by a similar process. The results will however be exhibited in a very different form.

We begin by taking a particular case of the equations in question,

$$\frac{d^m y}{dx^m} + ky = \frac{pm}{x}\,\frac{d^{m-1}y}{dx^{m-1}} \;\ldots\ldots\; (1),$$

where p is an integer.

Let $y = \Sigma a_n x^n$,

$$\therefore\; n(n-1)\ldots(n-m+2)(n-m+1-pm)\,a_n + k\,a_{n-m} = 0 \;\ldots\; (2).$$

Assume

$$a_n = \{n-m+1-(p-1)m\}\,\{n-m+1-(p-2)m\} \ldots (n-m+1) f(k)\, b_n \ldots\ldots (3),$$

$f(k)$ being some function of k, to be determined hereafter. Then

$$a_{n-m} = (n-m+1-pm)\,\{n-m+1-(p-1)\,m\} \ldots (n-m+1-m) f(k)\, b_{n-m} \ldots (4).$$

If we substitute these values in (2), every factor of (4) and every factor, except the last, of (3), will disappear, and the resulting equation will be

$$n(n-1)\ldots(n-m+2)(n-m+1)\,b_n + k\,b_{n-m} = 0 \ldots (5).$$

This is what (2) would be, were $p=0$. Hence $y=\Sigma\,.\,b_n x^n$ fulfils the equation $\dfrac{d^m y}{dx^m}+ky=0$, to which (1) would, in that case, be reduced. Let $y=\text{X}$ be the ordinary form of the solution of the last-written equation. Then we must obviously have

$$\Sigma\,.\,b_n x^n=\phi(k)\,.\,\text{X},$$

where $\phi(k)$ may be any function of k. A very little attention to the mode of integrating linear equations with constant coefficients will show, that in X, x always occurs in conjunction with $k^{\frac{1}{m}}$.

If we put
$$\text{X}=\Sigma\,.\,\text{A}_n x^n,$$
we must consequently have

$$\text{A}_n=\text{N}k^{\frac{n}{m}},$$

where N is a function of n, except for values of $n<m$, when it is an arbitrary constant; therefore

$$b_n=\text{N}\,\phi(k)\,k^{\frac{n}{m}}.$$

Recurring to (3), inverting the factors and multiplying and dividing by m^p, we shall easily deduce the following equation,

$$a_n=m^p\left(\frac{n-m+1}{m}\right)\left(\frac{n-m+1}{m}-1\right)$$
$$\dots\left(\frac{n-m+1}{m}-(p-1)\right)\text{N}\,.\,f(k)\,\phi(k)\,k^{\frac{n}{m}}.$$

The form of these p decreasing factors naturally suggests the idea of making $f(k)=k^{-p}$; and if we then put $\phi(k)=k^{-\frac{m-1}{m}}$, we get

$$a_n=m^p\left(\frac{n-m+1}{m}\right)\left(\frac{n-m+1}{m}-1\right)$$
$$\dots\left(\frac{n-m+1}{m}-(p-1)\right)\text{N}k^{\frac{n-m+1}{m}-p};$$

$$\therefore\ a_n=m^p\frac{d^p}{dk^p}b_n,\ \text{and}\ \therefore\ \Sigma a_n x^m=m^p\frac{d^p}{dk^p}\Sigma b_n x^n.$$

The factor m^p may obviously be neglected, and we shall therefore have, on replacing $\Sigma b_n x^n$ by $\phi(k)\text{X}$, *i.e.* by $\dfrac{\text{X}}{k^{\frac{m-1}{m}}}$, the following equation,

$$y=\frac{d^p}{dk^p}\,\frac{\text{X}}{k^{\frac{m-1}{m}}},$$

for the solution of (1), $y=\text{X}$ being that of $\dfrac{d^m y}{dx^m}+ky=0$.

If $m=2$, $X=C\sin\{\sqrt{(k)}\,x+\alpha\}$, and $\frac{m-1}{m}=\frac{1}{2}$. Hence

$$y=\frac{d^p}{dk^p}\,\frac{C\sin\{\sqrt{(k)}\,x+\alpha\}}{\sqrt{(k)}}$$

is the solution of

$$\frac{d^2y}{dx^2}+ky=\frac{2p}{x}\,\frac{dy}{dx}.$$

This result is given in *Hymers'* Diff. Equations, and is, I believe, due to Mr. Gaskin.

If the proposed equation were

$$\frac{d^my}{dx^m}+ky+\frac{pm}{x}\,\frac{d^{m-1}y}{dx^{m-1}}=0,$$

we should immediately conclude, from analogy, that its solution must be

$$y=\frac{d^{-p}}{dk^{-p}}\,\frac{X}{k^{\frac{m-1}{m}}};$$

but it may be as well to establish this conclusion by an independent investigation.

Equation (2) will, in this case, be

$$n(n-1)\ldots(n-m+2)(n-m+1+pm)\,a_n+k\,a_{n-m}=0.$$

Assume

$$a_n=f(k)\,\frac{b_n}{(n-m+1+pm)\{n-m+1+(p-1)m\}\ldots(n-m+1+m)},$$

$$\therefore\ a_{n-m}=f(k)\,\frac{b_{n-m}}{\{n-m+1+(p-1)m\}\ldots(n-m+1)},$$

$$\text{therefore}\quad n(n-1)\ldots(n-m+1)\,b_n+kb_{n-m}=0.$$

Therefore as before,

$$b_n=N\,.\,\varphi(k)\,k^{\frac{n}{m}},$$

$$\text{and}\quad a_n=m^{-p}N\,\frac{f(k)\,\phi(k)\,k^{\frac{n}{m}}}{\left(\frac{n-m+1}{m}+1\right)\cdots\left(\frac{n-m+1}{m}+p\right)}.$$

Here we make $f(k)=k^p$, and $\phi(k)=k^{-\frac{m-1}{m}}$,

$$\therefore\ a_n=m^{-p}N\,\frac{d^{-p}}{dk^{-p}}\,b_n,$$

no complementary function being added.

Hence, precisely as before, we find that

$$y=\frac{d^{-p}}{dk^{-p}}\,\frac{X}{k^{\frac{m-1}{m}}}$$

is the solution required.

Let us now consider the more general equation,

$$\frac{d^m y}{dx^m}+ky=pm\frac{d^{m-s-1}}{dx^{m-s-1}}\left(\frac{1}{x}\frac{d^s y}{dx^s}\right) \ldots\ldots (6).$$

If $y=\Sigma\,.\,a_n x^n$, there will be

$$n\ldots(n-s+1)(n-s-pm)(n-s-1)\ldots$$
$$\ldots(n-m+1)\,a_n+k\,a_{n-m}=0\ldots\ldots(7).$$

This equation is analogous to (2); but $m-1$ is replaced by s. Assume, therefore,

$$a_n=\{n-s-(p-1)m\}\{n-s-(p-2)m\}\ldots(n-s)\,k^{-p}.b_n\ldots(8),$$
$$\therefore\ a_{n-m}=(n-s-pm)\{n-s-(p-1)m\}\ldots\ldots(n-s+m)\,k^{-p}b_{n-m},$$

and $n\ldots(n-s+1)(n-s)(n-s-1)\ldots(n-m+1)b_n+k\,b_{n-m}=0.$

Hence, as before,

$$\Sigma b_n x^n=\phi(k)\,\mathrm{X},$$

where X denotes the same function of x that it did in the former case.

Consequently $\qquad b_n=\mathrm{N}\,\phi(k)\,k^{\frac{n}{m}}$, and

$$a_n=m^p\left(\frac{n-s}{m}\right)\left(\frac{n-s}{m}-1\right)\ldots\left\{\frac{n-s}{m}-(p-1)\right\}\mathrm{N}\,\phi(k)\,k^{\frac{n}{m}-p}.$$

We must, it is evident, make $\phi(k)=k^{-\frac{s}{m}}$, and then

$$a_n=m^p\frac{d^p}{dk^p}\mathrm{N}\,k^{\frac{n-s}{m}}=m^p\frac{d^p}{dk^p}b_n.$$

Hence $y=\dfrac{d^p}{dk^p}\dfrac{\mathrm{X}}{k^{\frac{s}{m}}}$ is the solution of the equation

$$\frac{d^m y}{dx^m}+ky=pm\frac{d^{m-s-1}}{dx^{m-s-1}}\left(\frac{1}{x}\frac{d^s y}{dx^s}\right),$$

for as before the factor m^p may be neglected.

A particular case of this result is that in which $s=0$. If with this value of s we have $m=2$, the equation to be integrated takes the form

$$\left(\frac{d^2 y}{dx^2}+ky\right)x^2=2p\left(x\frac{dy}{dx}-y\right),$$

and the solution is

$$y=\mathrm{C}\frac{d^p}{dk^p}\sin\{\surd(k)\,x+\alpha\}.$$

Equation (7) is the most general one in which the coefficient of a_n differs in one factor only from what it is in the case of

$$\frac{d^m y}{dx^m}+ky=0.$$

But our method is applicable in other cases.

Let us resume the equation discussed in the last number of the Journal,

$$\frac{d^m y}{dx^m} + ky = p(p-1)\frac{1}{x^2}\,\frac{d^{m-2}}{dx^{m-2}}\,y \;\ldots\ldots (8).$$

By the usual method of making $y = \Sigma a_n x^n$, we get

$$n(n-1)\,..(n-m+3)\,\{(n-m+2)\,(n-m+1)-p\,.\,(p-1)\}\,a_n + k\,a_{n-m} = 0,$$

or

$$n(n-1)\ldots(n-m+3)\,\{(n-m+2-p)\,(n-m+1+p)\}\,a_n + k\,a_{n-m} = 0 \ldots\ldots (9).$$

It may be remembered that we found it necessary that p or $p-1$ should be divisible by m. Suppose then that p is so divisible, and that the quotient is q.

Let

$$a_n = fk\,\frac{\{(n-m+2-(p-m)\}\,\{n-m+2-(p-2m)\}\ldots(n-m+2)}{(n-m+1+p)\,\{n-m+1+(p-m)\}\,\ldots\,(n-m+1+m)}\,b_n \;\ldots\ldots (10),$$

$$\therefore\; a_{n-m} = fk\,\frac{(n-m+2-p)\,\{n-m+2-(p-m)\}\ldots\ldots\ldots}{\{n-m+1+(p-m)\}\ldots\ldots\ldots(n-m+1)}\,b_{n-m},$$

(there are q factors in both numerator and denominator); equation (9) becomes

$$n(n-1)\,..(n-m+3)\,(n-m+2)\,(n-m+1)\,b_n + k\,b_{n-m} = 0;$$

and, as in the two preceding cases, we shall have

$$\Sigma\, b_n x^n = \phi(k)\,\mathrm{X},$$

$$\text{and}\qquad b_n = \mathrm{N}\,.\,k^{\frac{n}{m}}\,\phi(k).$$

(10) may be written thus, as $p = qm$,

$$a_n = f(k)\,\frac{\left(\frac{n-m+2}{m}\right)\ldots\ldots\left(\frac{n-m+2}{m}-q+1\right)}{\left(\frac{n-m+1}{m}+1\right)\ldots\ldots\left(\frac{n-m+1}{m}+q\right)}\,b_n.$$

Now

$$\frac{d^q}{dk^q}\,k^{\frac{n-m+2}{m}} = \left(\frac{n-m+2}{m}\right)\ldots\left(\frac{n-m+2}{m}-q+1\right)k^{\frac{n-m+2}{m}-q};$$

$$\text{therefore}\quad \frac{d^{-q}}{dk^{-q}}\left(k^{q-\frac{1}{m}}\,\frac{d^q}{dk^q}\,.\,k^{\frac{n-m+2}{m}}\right)$$

$$= \frac{\left(\frac{n-m+2}{m}\right)\ldots\ldots\left(\frac{n-m+2}{m}-q+1\right)}{\left(\frac{n-m+1}{m}+1\right)\ldots\ldots\left(\frac{n-m+1}{m}+q\right)}\,.\,k^{\frac{n-m+1}{m}+q}.$$

But if we make $\phi(k) = k^{\frac{-m+2}{m}}$, then

$$b_n = \mathrm{N} \,.\, k^{\frac{n-m+2}{m}}.$$

Let us also put $f(k) = k^{q-\frac{1}{m}}$; therefore

$$a_n = \frac{\left(\frac{n-m+2}{m}\right)\ldots\,\&\mathrm{c.}}{\left(\frac{n-m+1}{m}+1\right)\ldots\&\mathrm{c.}}\,\mathrm{N}\,k^{\frac{n-m+1}{m}+q}$$

$$= \mathrm{N}\,\frac{d^{-q}}{dk^{-q}}\left(k^{q-\frac{1}{m}}\,\frac{d^q}{dk^q}\,k^{\frac{n-m+2}{m}}\right),$$

$$\text{or}\quad a_n = \frac{d^{-q}}{dk^{-q}}\left(k^{q-\frac{1}{m}}\,\frac{d^q}{dk^q}\,b_n\right).$$

Hence
$$y = \frac{d^{-q}}{dk^{-q}}\left(k^{q-\frac{1}{m}}\,\frac{d^q}{dk^q}\,\frac{\mathrm{X}}{k^{\frac{m-2}{m}}}\right)\ \ldots\ldots\ \ldots\ldots\ (11),$$

is the solution required.

It admits also of another form, which it may be worth while to remark.

$$\frac{d^{-q}}{dk^{-q}}\,k^{\frac{n-m+1}{m}} = \frac{1}{\left(\frac{n-m+1}{m}+1\right)\ldots\ldots\left(\frac{n-m+1}{m}+q\right)}\,k^{\frac{n-m+1}{m}+q},$$

$$\text{therefore}\ \frac{d^q}{dk^q}\left(k^{-q+\frac{1}{m}}\,\frac{d^{-q}}{dk^{-q}}\,k^{\frac{n-m+1}{m}}\right)$$

$$= \frac{\left(\frac{n-m+2}{m}\right)\ldots\ldots\left(\frac{n-m+2}{m}-q+1\right)}{\left(\frac{n-m+1}{m}+1\right)\ldots\ldots\left(\frac{n-m+1}{m}+q\right)}\,k^{\frac{n-m+2}{m}-q}.$$

Here we must make $\phi(k)=k^{\frac{-m+1}{m}}$ and $f(k)=k^{-q+\frac{1}{m}}$, and then

$$b_n = \mathrm{N}\,k^{\frac{n-m+1}{m}}\ \text{and}\ a_n = \mathrm{N}\,(\ldots\ldots)\,k^{\frac{n-m+2}{m}-q},$$

$$\text{therefore}\ a_n = \frac{d^q}{dk^q}\left(k^{-q+\frac{1}{m}}\,\frac{d^{-q}}{dk^{-q}}\,b_n\right)$$

$$\text{and}\ y = \frac{d^q}{dk^q}\left(k^{-q+\frac{1}{m}}\,\frac{d^{-q}}{dk^{-q}}\,\frac{\mathrm{X}}{k^{\frac{m-1}{m}}}\right)\ \ldots\ldots\ldots\ (12).$$

The value of y, deduced from the developement of (12), can of course differ only in a factor of some function of k from that

which is given by (11), and it will easily appear, on comparing the values of a_n in the two cases, that this factor is $k^{-2q+\frac{1}{m}}$.

Let us now consider the case in which p is not divisible by m, while $p-1$ is so. And let $p-1=qm$. The two factors of (9) on which our reduction operates, viz.

$$(n-m+2-p)\,(n-m+1+p,)$$

may be written thus,

$$\{n-m+2+(p-1)\}\,\{n-m+1-(p-1)\},$$

$$\text{or}\quad (n-m+2+qm)\,(n-m+1-qm).$$

The change which this will introduce in the process, is not difficult to perceive. We must assume

$$a_n=f(k)\,\frac{\frac{n-m+1}{m}\cdots\cdots\frac{n-m+1}{m}-q+1}{\frac{n-m+2}{m}+1\cdots\cdots\frac{n-m+2}{m}+q}\,b_n$$

as before, the transformed equation will be

$$n\ldots\ldots(n-m+2)\,(n-m+1)\,b_n+k\,b_{n-m}=0.$$

Now

$$\frac{d^q}{dk^q}\,k^{\frac{n-m+1}{m}}=\left(\frac{n-m+1}{m}\right)\cdots\cdots\left(\frac{n-m+1}{m}-q+1\right)k^{\frac{n-m+1}{m}-q}$$

$$\text{therefore}\ \frac{d^{-q}}{dk^{-q}}\left(k^{q+\frac{1}{m}}\,\frac{d^q}{dk^q}\,k^{\frac{n-m+1}{m}}\right)$$

$$=\frac{\left(\frac{n-m+1}{m}\right)\cdots\cdots\left(\frac{n-m+1}{m}-q+1\right)}{\left(\frac{n-m+2}{m}+1\right)\cdots\cdots\left(\frac{n-m+2}{m}+q\right)}\,k^{\frac{n-m+2}{m}+q}.$$

If then we make $f(k)=k^{q+\frac{1}{m}}$ and $\phi(k)=k^{\frac{-m+1}{m}}$, we get

$$b_n=\mathrm{N}\,k^{\frac{n-m+1}{m}};\quad a_n=\mathrm{N}\,\frac{\left(\frac{n-m+1}{m}\right)\ldots\text{\&c.}}{\left(\frac{n-m+2}{m}+1\right)\ldots\text{\&c.}}\,k^{\frac{n-m+2}{m}+q},$$

$$\text{or}\ a_n=\mathrm{N}\,\frac{d^{-q}}{dk^{-q}}\left(k^{q+\frac{1}{m}}\,\frac{d^q}{dk^q}\,k^{\frac{n-m+1}{m}}\right).$$

Hence, finally,

$$y=\frac{d^{-q}}{dk^{-q}}\left(k^{q+\frac{1}{m}}\,\frac{d^q}{dk^q}\,\frac{\mathrm{X}}{k^{\frac{m-1}{m}}}\right)\ \ldots\ldots\ldots\ldots\ (13).$$

As an illustration, let us take the case of the equation which occurs in the theory of the figure of the earth,

$$\frac{d^2y}{dx^2} + ky = \frac{6y}{x^2}.$$

Here $m = 2$, $p = 3$, $p - 1 = 2$: hence $q = 1$, and as p is not a multiple of m, the formula (13) is to be used.

It is in this case, as $X = C \sin \{\sqrt{(k)}\,x + a\}$,

$$y = \frac{d^{-1}}{dk^{-1}} \left\{ k^{\frac{3}{2}} \frac{d}{dk} \frac{C \sin \{\sqrt{(k)}\,x + a\}}{k^{\frac{1}{2}}} \right\},$$

$$\frac{d}{dk} \frac{\sin \{\sqrt{(k)}\,x + a\}}{k^{\frac{1}{2}}} = \frac{1}{2} \frac{x}{k} \cos \{\sqrt{(k)}\,x + a\} - \frac{1}{2} \frac{\sin \{\sqrt{(k)}x + a\}}{k^{\frac{3}{2}}},$$

therefore

$$y = \tfrac{1}{2} C \frac{d^{-1}}{d\,k^{-1}} \left[x k^{\frac{1}{2}} \cos \{\sqrt{(k)}\,x + a\} - \sin \{\sqrt{(k)}\,x + a\}\right],$$

or integrating the first term by parts,

$$y = k C \sin \{\sqrt{(k)}\,x + a\} - \tfrac{3}{2} C \frac{d^{-1}}{dk^{-1}} \sin \{\sqrt{(k)}\,x + a\}.$$

But

$$\frac{d^{-1}}{dk^{-1}} \sin \{\sqrt{(k)}\,x + a\} = \frac{2k^{\frac{1}{2}}}{x} \cos \{\sqrt{(k)}\,x + a\} + \frac{2}{x^2} \sin \{\sqrt{(k)}\,x + a\},$$

therefore, if $kC = C_1$,

$$y = C_1 \left[\sin \{\sqrt{(k)}\,x + a\} + \frac{3}{x k^{\frac{1}{2}}} \cos \{\sqrt{(k)}\,x + a\} - \frac{3}{kx^2} \sin \{\sqrt{(k)}\,x + a\}\right],$$

which is the required solution.

Equation (13) corresponds to (11): but there is another form of the solution in the case of $p - 1 = qm$, which we shall just mention, and which is the counterpart of (12). It is

$$y = \frac{d^q}{dk^q} \left(k^{-q-\frac{1}{m}} \frac{d^{-q}}{dk^{-q}} \frac{X}{k^{\frac{m-2}{m}}} \right) \ldots\ldots\ldots (14).$$

It would be a needless repetition to go through the steps which lead to this result.

All the operations indicated in these symbolical solutions are practicable. This will appear by considering the nature of the function X, which, in its most general form, consists of the sum of terms, of which the type is

$$C\,e^{[a+\beta\sqrt{(-1)}]\,k^{\frac{1}{m}} x}.$$

If we make $k = \kappa^m$, this will become

$$C\, e^{[\alpha+\beta\sqrt{(-1)}]\,\kappa x},$$

which may be integrated any number of times for κ, and consequently, if it is multiplied by any rational and integral function of κ, it may still be integrated by parts as often as we please. Now $\frac{dk}{k^{\frac{s}{m}}}$ will become $m\kappa^{m-s-1}\,d\kappa$; and as m and s are integral, the method of parts applies, provided s is not greater than $m-1$, which it is in none of our formula.

Fourier's expression, by means of definite integrals for the i^{th} differential coefficient of any function, would enable us to extend our solutions to the cases in which p is fractional. But merely analytical transformations of the results at which we have arrived are not of much interest, and the methods of effecting them are direct and obvious.

Equation (1) admits of another symbolical solution besides the one already given.

It is easily seen, that if

$$\Sigma a_n x^n = x^{mp}\,\frac{d}{dx}\,\frac{1}{x^{m-1}}\cdot\frac{d}{dx}\,\frac{1}{x^{m-1}}\ldots\Sigma b_n x^n,\ (p \text{ factors}),$$

$$a_n = (n-pm+1)\,\ldots\ldots(n-m+1)\,b_n,$$

which is what (3) is, when $f(k) = 1$. ($\frac{d}{dx}$ applies to all that follows it.)

Hence we shall clearly have

$$y = x^{pm}\,\frac{d}{dx}\,\frac{1}{x^{m-1}}\,\frac{d}{dx}\,\frac{1}{x^{m-1}}\,\ldots\ldots\ldots\frac{d}{dx}\,\frac{1}{x^{m-1}}\,X,$$

for the solution of (1).

Similarly, the solution of (6) is

$$y = x^{(p-1)m+(s-1)}\cdot\frac{d}{dx}\,\frac{1}{x^{m-1}}\,\ldots\ldots\,\frac{d}{dx}\,\frac{1}{x^{s}}\,X.$$

Many applications and modifications of the method we have employed, will readily present themselves, but the subject is not of sufficient importance to deserve a fuller discussion. It is not difficult to multiply artifices, by means of which particular equations may be solved, but the results will, generally speaking, be of little value.

II.—ON A SIMPLE PROPERTY OF THE CONIC SECTIONS.*

The property in question is not new, though perhaps the proof here given may be so. I am principally induced to send it to the Journal as an illustration of the following remark.

The fundamental propositions of Geometry have each of them a train of consequences, any one of which might frequently be deduced from other fundamental propositions, but not so easily as from that, which we may therefore call its own. If then a very simple proposition will admit only of very complicated proof, it is open to trial whether the proper fundamental proposition has not been omitted from the elements.

The property of the conic sections above-mentioned is the following:—If the tangents at P and Q meet in the point T, and if S be one of the foci, PT and QT subtend equal angles at S, except only when P and Q are on different branches of an hyperbola; in which case the angles are supplemental. The simplicity of this proposition, compared with the difficulty of its proof by the ordinary properties of the curves, suggested the preceding remark, and on trial it appears that this property of the tangents of a conic section is the immediate consequence of the expression of the criterion that a circle touches four straight lines, which is omitted by Euclid.

Let there be four straight lines, no two of which are parallel. These must form such a figure as fig. 1, which may be considered as containing three four-sided figures, of which the sides are

(AB, BC, CD, DA), (AE, EC, CF, FA),
(BE, ED, DF, FB).

These may be called the convex, the single concave, and the double concave figures.

It is easily shown from the elements, that taking an angle of a triangle A, the circle which touches EA, AD, DE, is a function only of A, and of the excess of EA and AD above DE; while the circle which touches GE, ED, DH, is a function only of A, and of the sum of EA, AD, and DE. Hence the following

Theorem. A circle touches four straight lines if the sum of alternate sides be the same in the convex or the single concave figure, or if the sum of adjacent sides be the same in the double concave figure.

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* From a Correspondent.

Remark. Of the three equations

$$AB+CD=AD+BC, \quad AE+CF=AF+EC,$$
$$EB+BF=ED+DF,$$

any one being true, the others are necessarily true; and, any one of these being true, the circle which touches the four straight lines is inscribed in the convex figure; but the equation

$$BE+ED=BF+FD$$

shows that the circle is in the figure GECFH.

Let E and F be the foci of an ellipse, in which the points B and D lie; a circle can then be inscribed in ABCD, that is, the bisectors of the angles ABC and ADC meet in the same point as the bisectors of AED and AFB. But the first bisectors are the tangents at B and D; whence the property above enunciated follow:

Let E and F be the foci of an hyperbola, and A and C two points on the same branch. Then

$$AE+CF=AF+EC,$$

whence a circle can be inscribed in ABCD, the bisectors of A and C meet in the same point as those of E and F, and the same proposition is established.

Let E and F be the foci of an hyperbola, and B and D points on different branches. We have then

$$BE+ED=BF+FD;$$

a circle can be inscribed in GECFH, and the bisectors of EBF and EDF meet in the same point as those of GED and BFH, from whence it is easily shown that the angles subtended by the tangents at either focus are supplemental.

The case in which two lines of the fundamental figure become parallel, and its application to the parabola, offer no difficulty. One focus being then at an infinite distance, the property can only be obviously true of the other. But if through P, T, and Q, we draw lines parallel to the axis, and consider these lines as making infinitely small angles at the other focus, proportional to their distances from each other, we have the following theorem: If P and Q be two points of a parabola, the tangents to which meet in T, the projections of PT and QT upon any line perpendicular to the axis are equal to one another. This theorem can readily be proved by a more (geometrically) justifiable process.

A. D. M.

III.—ON THE MOTION OF A PENDULUM WHEN ITS POINT OF SUSPENSION IS DISTURBED.

In a former article* we investigated the nature of the mutual action of two pendulums united by any elastic or moveable connexion; we shall here consider more particularly the effect produced on the motion of a simple pendulum by a disturbance of its point of support. As before, we shall suppose the motions to be infinitesimal, in order that the equations may be at all manageable, and also for the sake of simplicity we shall assume the disturbances of the point of suspension to be rectilinear.

I. Let the point of suspension have a horizontal motion parallel to that of the pendulum.

The pendulum being a simple one, let u be the horizontal co-ordinate of the point of suspension, $u + x$ of the ball; let l be the length of the pendulum, and $n^2 = \frac{g}{l}$. Then the equation for the motion of the pendulum is

$$\frac{d^2(u + x)}{dt^2} + n^2 x = 0 \text{ .... } (1).$$

Two suppositions may be made regarding the nature of the disturbances of the point of suspension, that is, regarding the motion of u: either it has a vibratory motion independent of the motion of the pendulum or depending on it.

In the first case, when the motion of u is independent of that of the pendulum, u is given simply in terms of t, or

$$u = c \cos(at + \beta) \text{......} (2);$$

whence equation (1) becomes

$$\frac{d^2 x}{dt^2} + n^2 x - ca^2 \cos(at + \beta) = 0 \text{ ....} (3).$$

The solution of this is

$$x = \text{A} \cos(nt + \text{B}) - \frac{ca^2}{a^2 - n^2} \cos(at + \beta) \text{ ...} (4).$$

The motion, therefore, consists of two simple oscillations, which may be considered separately. The one expressed by the term $\text{A} \cos(nt + \text{B})$, is independent of the motion of the point of suspension, and depends only on the length of the pendulum and the original circumstances of the motion; the other, expressed by the term $\frac{ca^2}{a^2 - n^2} \cos(at + \beta)$, depends on the motion

* Vol. II. No. IX. p. 120.

of the point of suspension, and is synchronous with it, but both in extent and period is quite independent of the initial circumstances. If we neglect the first term, or the regular motion of the pendulum, we have for the disturbed motion

$$x = -\frac{ca^2}{a^2 - n^2}\cos(at + \beta),$$

$$\text{and} \quad u + x = -\frac{cn^2}{a^2 - n^2}\cos(at + \beta).$$

The motion of a point at a distance s from the ball will be

$$= \frac{c}{n^2 - a^2}\left(n^2 - a^2\frac{s}{l}\right)\cos(at + \beta).$$

That point, therefore, will remain at rest, so far as the disturbed motion is concerned, when

$$a^2 = \frac{l}{s}n^2 = \frac{g}{s},$$

or the distance from the ball of the point at rest, is equal to the length of a simple pendulum vibrating in the same time as the point of support, which might have been anticipated.

The preceding formulæ fail when $a = u$, or when the period of the disturbance is equal to that of the pendulum. In this case the integral of the equation (3) would be of the form

$$x = \text{A}\cos(nt + \beta) - \frac{ct}{2n}\sin(nt + \beta),$$

into which the time enters as a multiplier, so that x increases indefinitely with the time, and the motions are therefore no longer infinitesimal, as was at first supposed, and the original equations are therefore no longer applicable. This change of form in the integral is the analytical indication of a very important fact, viz. that a force, however small, may produce a motion of any extent in any body capable of oscillating, provided that the application of the force be made at intervals, the length of which is equal to the period in which the body would oscillate under the action of gravity. Thus it is, that a stone in a sling may be made to revolve completely round with a great angular velocity, merely by the synchronous motion of the hand; and many similar examples of this fact are constantly presented to our observation. We remember to have seen in a steam-vessel a lamp, which was so hung that its time of oscillation very nearly coincided with the stroke of the engine; the consequence of which was, that though the water was quite smooth, the lamp being set in motion by the reiterated strokes of the piston swung in a large arc, as if the vessel were rolling in a heavy sea.

In the second case, when the motion of the point of suspension depends partly on its own elasticity, and partly on the motion of the ball, we shall have for the equation of its motion

$$\frac{d^2u}{dt^2} + k^2u - ax = 0 \;\ldots\ldots\ldots\;\ldots \quad (5),$$

which, combined with the equation

$$\frac{d^2u}{dt^2} + \frac{d^2x}{dt^2} + n^2x = 0 \text{ ....} (1),$$

will determine u and x. To do this, muitiply (5) by $\frac{d^2}{dt^2}$ and (1) by $\left(\frac{d^2}{dt^2} + k^2\right)$, and subtract; then

$$\left\{\left(\frac{d^2}{dt^2} + k^2\right)\left(\frac{d^2}{dt^2} + n^2\right) + a\frac{d^2}{dt^2}\right\}x = 0,$$

$$\text{or} \quad \left(\frac{d^2}{dt^2} + \rho_1^2\right)\left(\frac{d^2}{dt^2} + \rho_2^2\right)x = 0 \text{ .. ...} (6),$$

ρ^2_1, ρ^2_2 being the roots of the quadratic equation

$$(\rho^2 - k^2)(\rho^2 - n^2) - a\rho^2 = 0 \text{ ....} (7).$$

Hence, integrating (6), we have for the value of x

$$x = \text{A} \cos(\rho_1 t + a) + \text{B} \cos(\rho_2 t + \beta) \text{ ...} (8).$$

To deduce the value of u, subtract (1) from (5); then

$$k^2u = \frac{d^2x}{dt^2} + (n^2 + a)x,$$

and therefore

$$u = \frac{\text{A}}{k^2}(n^2+a-\rho_1^2)\cos(\rho_1 t+a) + \frac{\text{B}}{k^2}(n^2+a-\rho_2^2)\cos(\rho_2 t+\beta) \text{...} (9).$$

In general k is much greater than n, and a is very small. The equation for determining the two values of ρ^2 may then be put under the forms

$$\rho_1^2 = n^2 - \frac{a\rho_1^2}{k^2 - \rho_1^2},$$

$$\rho_2^2 = k^2 + \frac{a\rho_2^2}{\rho_2^2 - n^2}.$$

One therefore of the two values of ρ will be a little less than n, and the other a little greater than k. Hence the signs of the coefficients of the first terms in x and u will be the same, and those of the second terms different.

The vibrations of the ball, and of the point of suspension, will thus consist of two parts, which may either co-exist or exist separately. The one part, the argument of which is cos $(\rho_1 t + a)$, is a synchronous vibration of the ball, and the point of suspension, a little slower than the independent motion of the ball, and such that the ball and the point of suspension are always on the same side of the perpendicular, passing through the original position. The other part, the argument of which is cos $(\rho_2 t + a)$, is a synchronous vibration, a little quicker than the independent vibration of the point of support, and such that the ball and the point of suspension are always on opposite sides of the perpendicular.

If instead of a simple pendulum we have a rod or other solid body, let a be the distance of its centre of gravity from the point of suspension, and k the radius of gyration; then if $u+x$ be the co-ordinate of the centre of gravity, and if we suppose the motion of the point of suspension to be independent of that of the pendulum, so that $u=c\cos(\alpha t+\beta)$, we shall have, for determining x, the equation

$$\frac{d^2x}{dt^2}+\frac{ga}{a^2+k^2}x-\frac{a^2}{a^2+k^2}c\alpha^2\cos(\alpha t+\beta)=0 \ldots\ldots (10).$$

Let $\frac{ga}{a^2+k^2}=n^2$. Then integrating

$$x=\mathrm{A}\cos(nt+\mathrm{B})-\frac{n^2\alpha^2ca}{g(\alpha^2-n^2)}\cos(\alpha t+\beta)\ldots\ldots(11).$$

Neglecting the first term, we get, as the expression for the part of the motion independent of the initial circumstances,

$$x=-\frac{n^2\alpha^2ca}{g(\alpha^2-n^2)}\cos(\alpha t+\beta).$$

For a point at a distance s from the point of suspension, the co-ordinate is $u+\frac{sx}{a}$

$$=\left(1-\frac{n^2}{g}\,\frac{\alpha^2s}{\alpha^2-n^2}\right)c\cos(\alpha t+\beta).$$

From this it appears that a point, the distance of which from the point of suspension is $\left(a+\frac{k^2}{a}\right)\left(1-\frac{n^2}{\alpha^2}\right)$ will remain at rest. If $\frac{n}{\alpha}=0$, that is, if α be very great, or the vibrations of the point of suspension become very rapid, the centre of percussion is the point which will remain at rest, as might have been anticipated.

II. Let the point of suspension have a horizontal motion at right angles to that of the pendulum.

Without going into the calculation in this case, it is easy to see that this disturbance will not affect the motion of the pendulum in its original direction, and that it will give rise to an oscillatory motion at right angles to that of the pendulum, the nature of which will be similar to the disturbance described in the first case. The two independent motions at right angles to each other will be combined into one, which will cause the ball of the pendulum to describe a curvilinear path.

III. Let the point of suspension have a small vertical oscillatory motion while the ball of the pendulum oscillates in one plane.

In this case, the disturbance of the motion of the pendulum is produced by the variation in the tension of the string; therefore, if

x represent the horizontal co-ordinate of the ball, and T the tension of the string, the equation of motion is

$$\frac{d^2x}{dt^2} + T\frac{x}{l} = 0 \ldots\ldots. (12).$$

Let the vertical motion of the point of suspension be

$$u = c\cos(at + \beta),$$

being independent of the motion of the pendulum. The ball receives the same motion from the change of tension, and therefore

$$\frac{d^2u}{dt^2} = -ca^2\cos(at+\beta) = T - g;$$

and therefore, putting n^2 for $\frac{g}{l}$, equation (12) becomes

$$\frac{d^2x}{dt^2} + n^2x - c\frac{a^2}{l}x\cos(at+\beta) = 0 \ldots\ldots (13).$$

This equation being no longer linear, cannot be integrated as the preceding equations were, and we must therefore have recourse to an approximate solution. If we suppose c to be small, we may substitute in that term the value of x derived from the supposition of $c=0$. That gives us

$$x = A\cos(nt + B),$$

and therefore

$$\frac{d^2x}{dt^2} + n^2x = \frac{Aca^2}{2l}[\cos\{(n+a)t + B + \beta\} + \cos\{(n-a)t + B - \beta\}].$$

Integrating this equation,

$$x = -\frac{Aca^2}{2l}\left\{\frac{\cos[(n+a)t+B+\beta]}{a^2+2an} + \frac{\cos[(n-a)t+B-\beta]}{a^2-2an}\right\} \ldots (14).$$

The most important conclusion from this result is, that if $A=0$, or the ball be originally at rest, it will have no motion communicated to it by the vertical motion of the point of suspension: and that if it has an oscillatory motion in any direction, this will not be permanently altered unless $a = 2n$, or the period of oscillation of the pendulum be double of that of the point of suspension. In this case the integral becomes infinite; and if, as before, we put it into another shape, we find that x increases continually with the time. This accords with experiment, which shows that the arc of vibration of a pendulum may be increased indefinitely by giving the point of suspension a vertical motion of oscillation, the period of which is half of that of the pendulum.

G. S.

IV.—ON THE INTEGRATION OF EQUATIONS OF PARTIAL DIFFERENTIALS.

By B. Bronwin.

It is only of linear equations of the second and higher orders that I propose to treat, and that for the sake of noticing the unnatural and preposterous nature of the received Theory with reference to them. For example, let

$$\frac{d^2z}{dx^2} + A\frac{d^2z}{dx\,dy} + B\frac{d^2z}{dy^2} = C;\ \frac{dz}{dx} = p,\ \frac{dz}{dy} = q.$$

From hence is derived

$$dp\,dy + Bdq\,dx - Cdx\,dy = \frac{d^2z}{dx\,dy}(dy^2 - Adx\,dy + Bdx^2).$$

And similarly when there are more variables, or the equation is of a higher order. Now to introduce the differentials of the second and higher degrees into quantities which are to be complete differentials, I call unnatural and preposterous. Next we make $dy = mdx$ to render the equation linear. But this is establishing a relation between two quantities absolutely independent, which is another absurdity. It is very true that in equations of the first order we do make such an assumption; but it is because we find a certain consequent to result from it. $dy - mdx = 0$ leads to $dM = 0$; and we want to find $M = a$. But this appears to me a very different thing from the way in which the assumption is made in the above equation.

The natural mode of treating the subject I conceive to be the following:

$$dp = \frac{d^2z}{dx^2}dx + \frac{d^2z}{dx\,dy}dy,\quad dq = \frac{d^2z}{dx\,dy}dx + \frac{d^2z}{dy^2}dy.$$

Multiply the last of these by the indeterminate m, add the product to the first, and to the sum the given equation multiplied by dx; we obtain

$$dp + mdq - Cdx = \frac{d^2z}{dx\,dy}\{dy + (m - A)\,dx\} + \frac{d^2z}{dy^2}(mdy - Bdx).$$

Now make

$$A - m = \frac{B}{m},\ \text{or}\ m^2 - Am + B = 0;$$

and let n and n' be the roots of this last; also

$$Q = \frac{d^2z}{dx\,dy} + n\frac{d^2z}{dy^2},\ \ Q' = \frac{d^2z}{dx\,dy} + n'\frac{d^2z}{dy^2},$$

and the above equation becomes

$$dp + ndq - Cdx = Q\,(dy - ndx);$$
$$\text{or}\quad dp + n'dq - Cdx = Q'\,(dy - n'dx).$$

Here we have only one indeterminate quantity, as in the received Theory, and the differentials are only of the first degree. If the first member be a complete differential, the second must be one also; and we may integrate as in an equation of the first order.

If the first member be not a complete differential, make

$$dM = \omega dN, \text{ or } M = \phi(N),$$

ω being an indeterminate, and M, N functions of x, y, z, p, and q. As x and y are independent, we must have

$$\left(\frac{dM}{dx}\right) = \omega\left(\frac{dN}{dx}\right), \quad \left(\frac{dM}{dy}\right) = \omega\left(\frac{dN}{dy}\right).$$

But, on account of the indeterminate ω, we must have

$$\left(\frac{dM}{dx}\right) = 0, \text{ \&c.};$$

$$\text{or } \frac{dM}{dp}\frac{dp}{dx} + \frac{dM}{dq}\frac{dq}{dx} + \frac{dM}{dz}p + \frac{dM}{dx} = 0;$$

$$\frac{dM}{dp}\frac{dp}{dy} + \frac{dM}{dq}\frac{dq}{dy} + \frac{dM}{dz}q + \frac{dM}{dy} = 0.$$

From the first of these we eliminate $\frac{dp}{dx}$ by the given equation, and there results

$$C\frac{dM}{dp} + \left(\frac{dM}{dq} - A\frac{dM}{dp}\right)\frac{dq}{dx} - B\frac{dM}{dp}\frac{dq}{dy} + \frac{dM}{dz}p + \frac{dM}{dx} = 0.$$

If now we multiply the second by the indeterminate m, and add the product to the equation last found; making $A - m = \frac{B}{m}$, or $m^2 - Am + B = 0$, of which the roots are n and n'; we obtain

$$\left(\frac{dq}{dx} + n'\frac{dq}{dy}\right)\left(\frac{dM}{dq} - n\frac{dM}{dp}\right) + C\frac{dM}{dp} + (p + n'q)\frac{dM}{dz} + n'\frac{dM}{dy} + \frac{dM}{dx} = 0.$$

As M must be free from the indeterminate quantities $\frac{dq}{dx}$, $\frac{dq}{dy}$, we must have

$$\frac{dM}{dq} - n\frac{dM}{dp} = 0.$$

$$C\frac{dM}{dp} + (p + n'q)\frac{dM}{dz} + n'\frac{dM}{dy} + \frac{dM}{dx} = 0.$$

We shall have two equations exactly like these for the determination of N. Any two particular solutions which satisfy both these equations may be taken for M and N.

Or assume $dp + ndq - Cdx = 0$, $dy - n'dx = 0$; and with these eliminate C, n, and n' from the above. There results

$$\frac{dM}{dp}dp + \frac{dM}{dq}dq + \frac{dM}{dz}dz + \frac{dM}{dy}dy + \frac{dM}{dx}dx = dM = 0.$$

In like manner we find $dN = 0$. These last therefore are simultaneous with the assumed equations; from which, if any how two complete differentials can be obtained, they are to be taken for $dM = 0$, $dN = 0$. And the integral of the proposed will be $M = \phi\,(N)$.

We will now take an equation of four variables.

$$\text{Let } \frac{d^2z}{dx^2} + A\frac{d^2z}{dy^2} + B\frac{d^2z}{du^2} + C\frac{d^2z}{dx\,dy} + D\frac{d^2z}{dx\,du} + E\frac{d^2z}{dy\,du} = F;$$

$$\frac{dz}{dx} = p, \quad \frac{dz}{dy} = q, \quad \frac{dz}{du} = r.$$

$$\text{Then } dp = \frac{d^2z}{dx^2}dx + \frac{d^2z}{dx\,dy}dy + \frac{d^2z}{dx\,du}du,$$

$$dq = \frac{d^2z}{dx\,dy}dx + \frac{d^2z}{dy^2}dy + \frac{d^2z}{dy\,du}du,$$

$$dr = \frac{d^2z}{dx\,du}dx + \frac{d^2z}{dy\,du}dy + \frac{d^2z}{du^2}du.$$

Adding all these, after multiplying the second by m, and the third by n, and subtracting the given equation multiplied by dx, we have

$$dp + mdq + ndr - Fdx = \frac{d^2z}{dy^2}(mdy - Adx) + \frac{d^2z}{du^2}(ndu - Bdx)$$

$$+ \frac{d^2z}{dx\,dy}\{dy - (C - m)\,dx\} + \frac{d^2z}{dx\,du}\{du - (D - n)\,dx\}$$

$$+ \frac{d^2z}{dy\,du}(mdu + ndy - Edx).$$

Make $\qquad C - m = \dfrac{A}{m}, \quad \text{or } m^2 - Cm + A = 0;$

$$D - n = \frac{B}{n}, \quad \text{or } n^2 - Dn + B = 0;$$

and let m, m' be the roots of the former, n, n' those of the latter; also make

$$m\frac{d^2z}{dy^2} + \frac{d^2z}{dx\,dy} = P, \quad n\frac{d^2z}{du^2} + \frac{d^2z}{dx\,du} = Q.$$

Then our equation becomes

$$dp + mdq + ndr - Fdx = P\,(dy - m'dx) + Q\,(du - n'dx)$$

$$+ \frac{d^2z}{dy\,du}(mdu + ndy - Edx).$$

$$\text{Or if } P + n\frac{d^2z}{dy\,du} = P', \quad Q + m\frac{d^2z}{dy\,du} = Q',$$

$$dp + mdq + ndr - Fdx = P'\,(dy - m'dx) + Q'\,(du - n'dx)$$

$$+ \frac{d^2z}{dy\,du}(mn' + m'n - E)\,dx.$$

This cannot be integrated unless $mn' + m'n - E = 0$, which is an equation of condition. If this be satisfied, then

$$dp + mdq + ndr - Fdx = P'(dy - m'dx) + Q'(du - n'dx);$$

which may be integrated as an equation of the first order.

Assume $M = \phi(N, T)$, or $dM = \omega dN + \omega' dT$. Then, as in the case of these variables,

$$\frac{dM}{dp}\frac{dp}{dx} + \frac{dM}{dq}\frac{dq}{dx} + \frac{dM}{dr}\frac{dr}{dx} + \frac{dM}{dz}p + \frac{dM}{dx} = 0,$$

$$\frac{dM}{dp}\frac{dp}{dy} + \frac{dM}{dq}\frac{dq}{dy} + \frac{dM}{dr}\frac{dr}{dy} + \frac{dM}{dz}q + \frac{dM}{dy} = 0,$$

$$\frac{dM}{dp}\frac{dp}{du} + \frac{dM}{dq}\frac{dq}{du} + \frac{dM}{dr}\frac{dr}{du} + \frac{dM}{dz}r + \frac{dM}{du} = 0.$$

Multiply the second by m', the third by n', and add the two products to the first. Subtract the given equation multiplied by $\frac{dM}{dp}$ from the sum, in order to eliminate $\frac{dp}{dx}$. Thus we have in virtue of the equations

$$m^2 - Cm + A = 0, \quad n^2 - Dn + B = 0;$$

$$\left(\frac{dq}{dx} + m'\frac{dq}{dy}\right)\left(\frac{dM}{dq} - m\frac{dM}{dp}\right) + \left(\frac{dr}{dx} + n'\frac{dr}{du}\right)\left(\frac{dM}{dr} - n\frac{dM}{dp}\right)$$
$$+ \frac{dq}{du}\left(n'\frac{dM}{dq} + m'\frac{dM}{dr} - E\frac{dM}{dp}\right) + F\frac{dM}{dp} + (p + m'q + n'r)\frac{dM}{dz}$$
$$+ \frac{dM}{dx} + m'\frac{dM}{dy} + n'\frac{dM}{du} = 0.$$

On account of the indeterminates remaining, we must have

$$F\frac{dM}{dp} + (p + m'q + n'r)\frac{dM}{dz} + \frac{dM}{dx} + m'\frac{dM}{dy} + n'\frac{dM}{du} = 0,$$

$$\frac{dM}{dq} - m\frac{dM}{dp} = 0, \quad \frac{dM}{dr} - n\frac{dM}{dp} = 0,$$

$$n'\frac{dM}{dq} + m'\frac{dM}{dr} - E\frac{dM}{dp} = 0.$$

The last by elimination gives $mn' + m'n - E = 0$, which is the equation of condition before found. We shall have equations exactly like them for the determination of N and T.

Now, if we suppose

$$dq + mdp + ndr - Fdx = 0,$$

$dy - m'dx = 0$, $du - n'dx = 0$; and by means of these eliminate F, m, m', n, and n' from the preceding; we again find $dM = 0$, and consequently $dN = 0$, $dT = 0$. These last therefore exist

simultaneously with the assumed equations; from which if we can any how obtain three integrals $M = a$, $N = b$, $T = c$, we shall have, for the integral of the proposed,

$$M = \varphi(N, T).$$

It is further to be observed that the quantities m and m', as also n and n' may change places, if by this means we can obtain other integrals of the first order.

We will next take an equation of the third order. Let

$$\frac{d^3z}{dx^3} + A\frac{d^3z}{dx^2\,dy} + B\frac{d^3z}{dx\,dy^2} + C\frac{d^3z}{dy^3} = D;$$

$$\text{and let}\quad \frac{d^2z}{dx^2} = p', \quad \frac{d^2z}{dx\,dy} = q', \quad \frac{d^2z}{dy^2} = r'.$$

Then $dp' = \dfrac{d^3z}{dx^3}dx + \dfrac{d^3z}{dx^2\,dy}dy$, $dq' = \dfrac{d^3z}{dx^2\,dy}dx + \dfrac{d^3z}{dx\,dy^2}dy$,

$$dr' = \frac{d^3z}{dx\,dy^2}dx + \frac{d^3z}{dy^3}dy.$$

Adding all these, after multiplying the second by m and the third by n, and eliminating $\dfrac{d^3z}{dx^3}$ by the given equation; we obtain

$$dp' + mdq' + ndr' - Ddx = \frac{d^3z}{dx^2\,dy}\{dy - (A - m)dx\}$$

$$+ \frac{d^3z}{dx\,dy^2}\{mdy + (n - B)\,dx\} + \frac{d^3z}{dy^3}(ndy - Cdx).$$

Make $A - m = \dfrac{C}{n}$, $\dfrac{B - n}{m} = \dfrac{C}{n}$. By eliminating m, we find

$$n^3 - Bn^2 + ACn - C^2 = 0.$$

Let the roots of this be n, n', and n''; and let m, m', and m'' be the corresponding values of m. Our equation now becomes

$$dp' + mdq' + ndr' - Ddx = Q\left(dy - \frac{C}{n}dx\right);$$

where Q is a function of $\dfrac{d^3z}{dx^2\,dy}$, &c., and the differentials are only of the first degree.

Suppose $M = \varphi(N)$ the integral of this. As before, we shall have the partial differentials of each member separately equal to nothing. Or

$$\frac{dM}{dp'}\frac{dp'}{dx} + \frac{dM}{dq'}\frac{dq'}{dx} + \frac{dM}{dr'}\frac{dr'}{dx} + \frac{dM}{dp}\frac{dp}{dx} + \frac{dM}{dq}\frac{dq}{dx} + \frac{dM}{dz}p + \frac{dM}{dx} = 0,$$

$$\frac{dM}{dp'}\frac{dp'}{dy} + \frac{dM}{dq'}\frac{dq'}{dy} + \frac{dM}{dr'}\frac{dr'}{dy} + \frac{dM}{dp}\frac{dp}{dy} + \frac{dM}{dq}\frac{dq}{dy} + \frac{dM}{dz}q + \frac{dM}{dy} = 0.$$

If we eliminate $\frac{dp'}{dx}$ from the first by the given equation, and add the result to the second multiplied by $\frac{\text{C}}{n}$; we obtain

$$\left(\frac{dq'}{dx}+\frac{\text{C}}{n}\frac{dq'}{dy}\right)\left(\frac{d\text{M}}{dq'}-m\frac{d\text{M}}{dp'}\right)+\left(\frac{dr'}{dx}+\frac{\text{C}}{n}\frac{dr'}{dy}\right)\left(\frac{d\text{M}}{dr'}-n\frac{d\text{M}}{dp'}\right)$$
$$+\left(p'+\frac{\text{C}}{n}q'\right)\frac{d\text{M}}{dp}+\left(q'+\frac{\text{C}}{n}r'\right)\frac{d\text{M}}{dq}+\text{D}\frac{d\text{M}}{dp'}+\left(p+\frac{\text{C}}{n}q\right)\frac{d\text{M}}{dz}$$
$$+\frac{d\text{M}}{dx}+\frac{\text{C}}{n}\frac{d\text{M}}{dy}=0.$$

Whence, on account of the indeterminate quantities remaining, we must have

$$\frac{d\text{M}}{dq'}-m\frac{d\text{M}}{dp'}=0,\quad \frac{d\text{M}}{dr'}-n\frac{d\text{M}}{dp'}=0,$$
$$\left(p'+\frac{\text{C}}{n}q'\right)\frac{d\text{M}}{dp}+\left(q'+\frac{\text{C}}{n}r'\right)\frac{d\text{M}}{dq}+\text{D}\frac{d\text{M}}{dp'}+\left(p+\frac{\text{C}}{n}q\right)\frac{d\text{M}}{dz}$$
$$+\frac{d\text{M}}{dx}+\frac{\text{C}}{n}\frac{d\text{M}}{dy}=0.$$

Now assume $dp'+mdq'+ndr'-\text{D}dx=0,\ dy-\frac{\text{C}}{n}dx=0$: and by these eliminate C, D, m and n from the preceding equations, and we again find $d\text{M}=0$, and consequently also $d\text{N}=0$. If then from the assumed equations we can find two integrals $\text{M}=a$, $\text{N}=b$; we shall have $\text{M}=\phi(\text{N})$ for an integral of the proposed of the second order. The two other values of m and n will give two other integrals $\text{M}'=\phi(\text{N}')$, $\text{M}''=\chi(\text{N}'')$, when it is practicable to find all three.

In the preceding theory the differentials never rise above the first degree, and they ought not. The method in fact is similar to that employed upon equations of the first order; and it must be evident that it is the natural and proper one. Those who first treated on this subject must have overlooked it from not proceeding in the most simple manner, or from not perceiving how they could eliminate a sufficient number of the indeterminates or differential coefficients.

Denby, near Wakefield.

V.—ON THE EVALUATION OF A DEFINITE MULTIPLE INTEGRAL.

By D. F. Gregory, B.A., Fellow of Trinity College.

In a memoir read before the Academy of Sciences of Paris, and inserted in the *Comptes rendus,* Vol. VIII. p. 156, M. Lejeune Dirichlet called the attention of mathematicians to the remarkable multiple integral

$$V = \int dx \int dy \int dz \ldots\ldots x^{a-1}\, y^{b-1}\, z^{c-1} \ldots\ldots (1),$$

which is to be taken between the positive limits of the variables determined by the inequality

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots \mathrel{\substack{=\\<}} 1 \ldots\ldots\ldots (2),$$

the number of variables, and therefore of integrals, being any whatever. The result at which M. Dirichlet arrives is, that

$$V = \frac{\alpha^a \beta^b \gamma^c \ldots}{pqr \ldots} \, \frac{\Gamma\left(\frac{a}{p}\right) \Gamma\left(\frac{b}{q}\right) \Gamma\left(\frac{c}{r}\right) \ldots}{\Gamma\left(1 + \frac{a}{p} + \frac{b}{q} + \frac{c}{r} \ldots\right)},$$

Γ being the second Eulerian Integral.

The actual calculation is not given in the paper referred to, though the process is indicated; but M. Liouville has investigated the value of the integral by a method different from that employed by M. Dirichlet, and his memoir (*Journal de Mathématiques,* Vol. IV. p. 225) is a very elegant specimen of analysis. The integral itself deserves attention, not only as being a remarkable analytical extension of that property of the first Eulerian Integral by which it is connected with the second, but also because it frequently occurs in the investigation of areas of curves, contents of solids, centres of gravity, and other physical and geometrical problems of a similar kind. From its extensive application to cases which are of such frequent occurrence, this multiple integral ought to receive a prominent place in elementary works on the Integral Calculus, and on that account I here bring it before the English reader. The method by which I propose to evaluate this integral is, I believe, new; and I am anxious to show its application in this case, not only because it exhibits very distinctly the nature of the connexion between this integral and the function Γ, but because it can also be applied with great advantage to the calculation of a number of other definite integrals. In the present paper, however, I shall confine myself to the integral of M. Dirichlet, and a more general one of the same kind which is given by M. Liouville.

In the first place, following the method of M. Liouville, we shall

transform the integral so that the limits shall be of the first degree only. This is easily done by assuming

$$\left(\frac{x}{\alpha}\right)^p = x', \quad \left(\frac{y}{\beta}\right)^q = y', \quad \left(\frac{z}{\gamma}\right)^r = z', \text{ \&c.}$$

from which

$$dx = \frac{\alpha}{p} x'^{\frac{1}{p}-1}\, dx', \quad dy = \frac{\beta}{q} y'^{\frac{1}{q}-1}\, dy', \quad dz = \frac{\gamma}{r} z'^{\frac{1}{r}-1}\, dz', \text{ \&c.}$$

On substituting these values for the variables and their differentials, the integral becomes

$$V = \frac{\alpha^a \beta^b \gamma^c \dots}{pqr \dots} U \dots\dots\dots (3),$$

where U is the definite integral

$$\int dx' \int dy' \int dz' \dots x'^{\frac{a}{p}-1} y'^{\frac{b}{q}-1} z'^{\frac{c}{r}-1} \dots (4),$$

the limits of the variables being given by the inequality

$$x' + y' + z' \dots\dots \overline{\overline{<}}\ 1 \dots\dots\dots\dots\dots (5).$$

Now let

$$\frac{a}{p} = l, \quad \frac{b}{q} = m, \quad \frac{c}{r} = n \dots\dots\ ..$$

and, dropping the accents which are no longer necessary for discrimination, we have to calculate the integral

$$U = \int dx \int dy \int dz \dots\dots x^{l-1} y^{m-1} z^{n-1} \dots\dots (6),$$

the limits being given by the inequality

$$x + y + z \dots\dots \overline{\overline{<}}\ 1.$$

If the variables be only two in number, x and y, the integral is reduced to

$$U = \int dx \int dy\, x^{l-1} y^{m-1} = \frac{1}{m} \int dx\, x^{l-1} (1-x)^m,$$

since the limits of y are 0 and $1 - x$.

The evaluation of this integral, by a method due to Professor Jacobi, may be found in this Journal, Vol. I., p. 94, and it is by an extension of that method that M. Liouville has calculated the general integral under consideration. Instead of employing it we shall proceed in the following manner.

Let $\qquad x + y + z + \dots\dots\dots = v,$

$\qquad$ or $\quad x = v - y - z - \dots\dots$

Then, as x varies when y, z ..... are constant, $dx = dv$, and (10) becomes

$$U = \int dv \int dy \int dz\, y^{m-1} z^{n-1} (v - y - z \dots)^{l-1} \dots\dots\dots (7).$$

We might now integrate with respect to v but, for the conveni-

ence of our future operations, we shall only indicate the operations. The extreme limits of v are 0 and 1, and we may therefore write

$$U=\int_0^1 dv \int dy \int dz\, y^{m-1} z^{n-1} (v-y-z-\ldots)^{l-1}.$$

Now by the symbolical form of Taylor's Theorem we have

$$f(x+h) = \varepsilon^{h\frac{d}{dx}} f(x).$$

Hence we may put

$$(v-y-z-\ldots)^{l-1} = \varepsilon^{-y\frac{d}{dv}} (v-z-\ldots)^{l-1} \ldots\ldots (9),$$

and we have then

$$U=\int_0^1 dv \int dz\, z^{n-1} \ldots \int dy\, y^{m-1}\, \varepsilon^{-y\frac{d}{dv}} (v-z-\ldots)^{l-1} \ldots\ldots (10),$$

the limits of y being 0 and $v - z - \ldots$.

Now assume $y\dfrac{d}{dv} = t$, so that $dy = dt\left(\dfrac{d}{dv}\right)^{-1}$,

$$U=\int_0^1 dv \int dz\, z^{n-1} \int dt\, t^{m-1}\, \varepsilon^{-t} \left(\frac{d}{dv}\right)^{-m} (v-z\ldots)^{l-1} \ldots (11).$$

To find the limits of t we have recourse to the following considerations. Supposing, for simplicity, that there were only two variables, x and y, we have

$$(v-y)^{l-1} = \varepsilon^{-y\frac{d}{dv}}\, v^{l-1} = \varepsilon^{-t} \cdot v^{l-1}.$$

Now, when $y = 0$ the first side becomes v^{l-1}; and in order that the second side should be reduced to that form, we must have $t = 0$. Again, when $y = v$ the first side becomes zero, l being positive; and in order that the second side may also become zero, we must have $t = \infty$.

Hence to the values

$$\left.\begin{matrix} y = 0 \\ y = v \end{matrix}\right\} \text{ correspond } \left\{\begin{matrix} t = 0 \\ t = \infty. \end{matrix}\right.$$

As in this transformation consists the principle of the method I employ, I shall add a few words in explanation of it. When we assume

$$y\frac{d}{dv} = t \text{ and } dy = dt\left(\frac{d}{dv}\right)^{-1},$$

and therefore

$$\varepsilon^{-y\frac{d}{dv}}\, v^{l-1} = \varepsilon^{-t}\, v^{l-1},$$

we suppose t to be a variable capable of increase or decrease; or, in other words, a symbol of quantity. But, on the other hand, $y\dfrac{d}{dv}$ is a symbol of operation to which we cannot apply the terms increase or decrease, and it may therefore seem to be scarcely allowable to assume it to be equal to a quantitative symbol It is to be

observed, however, that our use of the symbolical expression is only for the assumption of the *form* of the new function, and that when we put $y\dfrac{d}{dv}=t$, we really say nothing more than that t is to be such a function of y that it shall satisfy the equation

$$(v-y)^{l-1}=\varepsilon^{-t}v^{l-1},$$

which of course is an assumption which we are quite at liberty to make. Whether in every case of such a transformation we could actually determine t, is a question with which we have fortunately no concern, as it is to be expected that such a determination would be in most cases very difficult if not impracticable. All that we require to know are the limiting values of t corresponding to those of y; and as these can generally be found by considerations such as we have used, the transformation is for our purposes sufficient. It reduces the function to be integrated to a very convenient form for effecting that operation, and the substitution of the values of the limits offers generally no difficulties. What we have said regarding the limits in the case of two variables, applies equally well to a greater number, except that the limits of y being 0 and $v-z-\ldots$, the limits of t will still be 0 and ∞. Hence, recurring to our integral, equation (11) becomes, on affixing the limits to t,

$$\mathrm{U}=\int_0^1 dv\int dz\, z^{n-1}\ldots\int_0^\infty dt\, t^{m-1}\varepsilon^{-t}\left(\frac{d}{dv}\right)^{-m}(v-z-\ldots)^{l-1}.$$

Now $\int_0^\infty dt\, t^{m-1}\varepsilon^{-t}=\Gamma(m)$, and therefore

$$\mathrm{U}=\Gamma(m)\int_0^1 dv\int dz\, z^{n-1}\ldots\left(\frac{d}{dv}\right)^{-m}(v-z-\ldots)^{l-1}\ \ldots\ldots (12).$$

Proceeding with z in the same manner as with y, by putting

$$(v-z-\ldots)^{l-1}=\varepsilon^{-z\frac{d}{dv}}v^{l-1},$$

and assuming

$$z\frac{d}{dv}=s,$$

we find, as before,

$$\mathrm{U}=\Gamma(m)\int_0^1 dv\ldots\int_0^\infty ds\, s^{n-1}\varepsilon^{-s}\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1},$$

$$=\Gamma(m)\,\Gamma(n)\int_0^1 dv\ldots\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1}\ldots\ldots(13).$$

In this manner we might proceed for any number of variables, but restricting ourselves to three, it only remains to integrate with respect to v between the given limits.

Now as

$$\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1}=\frac{\Gamma(l)}{\Gamma(l+m+n)}v^{l+m+n-1},$$

we find

$$U = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_0^1 dv\, v^{l+m+n-1}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{(l+m+n)\,\Gamma(l+m+n)}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(1+l+m+n)} \quad \ldots\ldots\ \ldots\ldots\ldots \ (14);$$

since by the fundamental property of the function Γ

$$(l+m+n)\,\Gamma(l+m+n) = \Gamma(1+l+m+n).$$

Substituting this in (3) and putting for l, m, n, their values $\frac{a}{p}$, $\frac{b}{q}$, $\frac{c}{r}$, we finally obtain

$$V = \frac{\alpha^a \beta^b \gamma^c}{pqr} \; \frac{\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{b}{q}\right)\Gamma\left(\frac{c}{r}\right)}{\Gamma\left(1+\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)} \quad \ldots.\ \ldots(15).$$

It is to be observed, that in effecting the operation

$$\left(\frac{d}{dv}\right)^{-(m+n)} v^{l-1},$$

we must not add any arbitrary constants, since this inverse operation has arisen during the process without causing any constants to disappear, and there are therefore none to be restored. All the arbitrary constants arising from the original integrals are eliminated in taking the limits, and no others are to be introduced, otherwise we should have more arbitrary constants than integrals.

The transformation in (11) and the subsequent investigation of the limits are the parts of this method which appear to be new, or at least not to have been hitherto employed to calculate definite integrals. It is easily seen that the same transformation may be applied to many other definite integrals, but it would occupy too much space to enter on the consideration of them at present. I shall therefore pass on to some examples of the application of the formula which has just been proved.

Ex. 1. To find the area of the evolute to the ellipse.

The expression for the area is

$$V = \iint dx\, dy,$$

x and y being subject to the limiting condition

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

Here $a = 1$, $b = 1$, $p = q = \frac{2}{3}$. Therefore by (15)

$$V = \frac{9}{4}\,\alpha\beta\,\frac{\{\Gamma(\frac{3}{2})\}^2}{\Gamma(4)}.$$

Now $\Gamma(4)=3.2.1$, and $\Gamma(\frac{3}{2})=\frac{1}{2}\Gamma(\frac{1}{2})=\frac{1}{2}\sqrt{\pi}$,

and therefore $V=\frac{3\alpha\beta}{32}$, which is the area of the portion of the curve included between the positive axes, since the variables are supposed never to become negative. The area of the whole curve is $\frac{3\alpha\beta}{8}$.

Ex. 2. If we wish to find the co-ordinates of the centre of gravity of the same area, we have to calculate

$$\iint x\,dx\,dy \text{ and } \iint y\,dx\,dy.$$

By the formula (15)

$$\iint x\,dx\,dy=\frac{9}{4}\alpha^2\beta\,\frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(1+3+\frac{3}{2})},\quad \iint y\,dx\,dy=\frac{9}{4}\alpha\beta^2\,\frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(1+3+\frac{3}{2})}.$$

Hence if $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre of gravity,

$$\bar{x}=\frac{2^8\alpha}{9.7.5},\quad \bar{y}=\frac{2^8\beta}{9.7.5}.$$

Ex. 3. It is shown in this Journal (Vol. II p. 14,) that the equation to the parabola, when referred to two tangents as axes, is

$$\sqrt{\frac{x}{\alpha}}+\sqrt{\frac{y}{\beta}}=1,$$

α and β being the portion of the tangents intercepted between their intersection and the curve. If θ be the angle between the axes, the area is

$$V=\sin\theta\iint dx\,dy,$$

the limits of x and y being given by the preceding equation. Here $a=1$, $b=1$, $p=q=\frac{1}{2}$. Therefore

$$V=\sin\theta\,.\,4\alpha\beta\,\frac{\{\Gamma(2)\}^2}{\Gamma(5)}=\frac{\alpha\beta\sin\theta}{6}.$$

From this it appears (referring to the figure in the article alluded to above,) that the triangle ABC is three times the area ABPC.

Ex. 4. To find the centre of gravity of the eighth part of the ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1.$$

If $\bar{z}$ be one of the co-ordinates of the centre of gravity,

$$\bar{z}=\frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

the limits being given by the preceding equation.

In the numerator, comparing it with (15), we have

$$a=b=1,\quad c=2,\quad p=q=r=2.$$

Therefore

$$\iiint z\, dx\, dy\, dz = \frac{\alpha\beta\gamma^2}{8}\,\frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(3)} = \pi\frac{\alpha\beta\gamma^2}{16}.$$

In the denominator $a=b=c=1$, $p=q=r=2$. Therefore

$$\iiint dx\, dy\, dz = \frac{\pi\alpha\beta\gamma}{8}\,\frac{\{\Gamma(\frac{1}{2})\}^3}{\Gamma(1+\frac{3}{2})} = \frac{\pi\alpha\beta\gamma}{6}.$$

Hence $\bar{z} = \frac{3}{8}\gamma$, and similarly for the other co-ordinates.

M. Liouville has given to the Theorem of M. Dirichlet a very important extension, of which the following is the enunciation. If

$$W = \int dx \int dy \int dz \ldots x^{l-1} y^{m-1} z^{n-1} \ldots f(x+y+z+\ldots) \quad (16),$$

where the limits of $x, y, z \ldots$ are such as to satisfy the inequality

$$x+y+z+\ldots \overline{\overline{<}}\, h,$$

f being any function whatsoever,

$$W = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\ldots}{\Gamma(l+m+n+\ldots)} \int_0^h dv\, f(v)\, v^{l+m+n+\ldots-1} \ldots\ldots (17).$$

It will be seen, as in the first part of this article, that to the form (16) may be reduced the more general one

$$W' = \int dx \int dy \int dz \ldots x^{a-1}\, y^{b-1}\, z^{c-1} f\left\{\left(\frac{x}{a}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots\right\},$$

where the limiting values are given by the inequality

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots\ldots \overline{\overline{<}}\, h;$$

for by a simple transformation of the variables we should find

$$W' = \frac{a^a \beta^b \gamma^c}{pqr}\, W,$$

and it is therefore only necessary to calculate W. To effect this we proceed in the same manner as before.

Let $\qquad x + y + z + \ldots = v,$

then $\quad dx = dv,$

the extreme limits of v being 0 and h. Then

$$W = \int_0^h dv \int dy \int dz \ldots y^{m-1}\, z^{n-1} \ldots (v-y-z-\ldots)^{l-1} f(v).$$

Now, as before,

$$(v-y-z-\ldots)^{l-1} f(v) = \varepsilon^{-y\frac{d'}{dv}} (v-z-\ldots)^{l-1} f(v),$$

where $\frac{d'}{dv}$ is accentuated to imply that it refers only to the v included in $(v-z-\ldots)^{l-1}$ and not to that under the f. Now putting $y\frac{d'}{dv} = t$, the limits of t will be 0 and ∞, and

$$W=\int_0^h dv \int dz \ldots z^{n-1} \ldots \int_0^\infty dt\, t^{m-1}\, \varepsilon^{-t} \left(\frac{d'}{dv}\right)^{-m} (v-z-\ldots)^{l-1} f(v),$$

$$=\Gamma(m) \int_0^h dv \int dz \ldots z^{n-1} \ldots \left(\frac{d'}{dv}\right)^{-m} (v-z-\ldots)^{l-1} f(v).$$

Next, treating z in the same manner as y, we have

$$W=\Gamma(m)\,\Gamma(n) \int_0^h dv \ldots \left(\frac{d'}{dv}\right)^{-(m+n)} (v-\ldots)^{l-1} f(v),$$

and so on for any number of variables. Restricting ourselves to three, and effecting the operation $\left(\frac{d'}{dv}\right)^{-(m+n)}$, which has reference only to v^{l-1}, we find

$$W=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_0^h dv\, v^{l+m+n-1} f(v).$$

As an example of the application of this formula, let us take the expression

$$W=\int dx \int dy \int dz \frac{1}{\sqrt{1-x^2-y^2-z^2}},$$

where the variables satisfy the condition

$$x^2+y^2+z^2 \overline{\overline{<}}\, 1.$$

To change the variables, put $x^2=x'$, $y^2=y'$, $z^2=z'$; then

$$W=\tfrac{1}{8} \iiint \frac{dx'\,dy'\,dz'}{\sqrt{x'y'z'}} \frac{1}{\sqrt{1-x'-y'-z'}},$$

$$\text{and } x'+y'+z' \overline{\overline{<}}\, 1.$$

In this case, then, $l=m=n=\frac{1}{2}$; therefore

$$W=\frac{1}{8} \frac{\{\Gamma(\frac{1}{2})\}^3}{\Gamma(\frac{3}{2})} \int_0^1 \frac{v^{\frac{1}{2}}dv}{\sqrt{1-v}};$$

putting $v=x^2$, we find

$$\int_0^1 \frac{v^{\frac{1}{2}}dv}{\sqrt{1-v}} = 2\int_0^1 \frac{x^2dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\pi.$$

Also $\Gamma(\frac{1}{2})=\sqrt{\pi}$, and $\Gamma(\frac{3}{2})=\frac{1}{2}\sqrt{\pi}$; therefore

$$W=\frac{\pi^2}{8}.$$

Again, take $W=\iint dx\,dy \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}$,

$$\text{when } x^2+y^2 \overline{\overline{<}}\, 1.$$

By a change of the variable,

$$W=\tfrac{1}{4}\iint \frac{dx\,dy}{\sqrt{xy}} \sqrt{\frac{1-x-y}{1+x+y}}, \quad \text{and } x+y \overline{\overline{<}}\, 1.$$

Here $l=m=\frac{1}{2}$; therefore

$$W=\frac{1}{4}\frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(1)}\int_0^1 dv\sqrt{\frac{1-v}{1+v}}.$$

Now $\Gamma(1)=1$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$,

$$\int_0^1 dv\sqrt{\frac{1-v}{1+v}}=\int_0^1 dv\left(\frac{1}{\sqrt{1-v^2}}-\frac{v}{\sqrt{1-v^2}}\right)=\frac{\pi}{2}-1;$$

$$\text{therefore}\quad W=\frac{\pi}{4}\left(\frac{\pi}{2}-1\right).$$

Other examples of this formula will be found in a paper by M. E. Catalan, *Journal de Mathématiques*, Vol. IV. p. 323.

VI.—REMARKS ON POISSON'S PROOF OF THE PROPOSITION THAT $F(\mu.\omega)$ MAY BE EXPANDED IN A SERIES OF LAPLACE'S COEFFICIENTS.*

I propose in the following paper to simplify the integration which occurs in the proof of this proposition as given by Poisson (*Théorie de la Chaleur*, Art. 106), and by Pratt (*Mechanics*, Art. 176), and to allude to a point which appears to present some difficulty in the articles just referred to.

The definite integral which occurs is

$$\int_{-1}^1\int_0^{2\pi}\frac{(1-c^2)\,d\mu'\,d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}}\ (=\text{X suppose}),$$

$\{$where $p=\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')\}$ when $c=1$.

It is manifest that the element of the integral will have no sensible value unless $p=1$; that is, unless $\mu'=\mu$ and $\omega'=\omega+$ some multiple of 2π. Let then the integral be transformed, as in Pratt, and we have

$$X=\iint\frac{2g\,d\nu\,dz}{\left\{g^2+\dfrac{\nu^2}{1-\mu^2}+z^2(1-\mu^2)\right\}^{\frac{3}{2}}}$$

$$=\iint\frac{2g\,d\nu'\,dz'}{(g^2+\nu'^2+z'^2)^{\frac{3}{2}}},$$

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* From a Correspondent.

$$\text{if } \nu' = \frac{\nu}{\sqrt{1-\mu^2}}, \quad z' = z\sqrt{1-\mu^2}.$$

Now for distinctness take the rectangle $acdb$, (fig. 2),

$$\text{A}a = \text{A}c = 1, \quad \text{AB} = 2\pi.$$

Take A as origin, and let values of ω' be represented by lines parallel to AB, values of μ' by lines perpendicular to it. Then, since the limits of μ' are $+1$ and -1, and those of ω' 2π and 0, all points whose co-ordinates are corresponding values of ω' and μ' will lie within $acdb$. And if we take P as a point represented by the values of ω and μ, the above integration will be effected by integrating through any small space round P, since it is only at P that the elements are not evanescent.

These considerations of space suggest a simplification of the integral, viz. by trausforming it to polar co-ordinates r and θ: which being done, we have

$$\text{X} = \iint \frac{2grd\theta dr}{(g^2+r^2)^{\frac{3}{2}}}.$$

Now suppose,

I. That μ and ω have not the values ± 1 or 2π and 0, so that P lies within $acdb$. Then we must integrate through a small space round P, and the limits will therefore be 0 and 2π for θ, and 0 and any small value (a) for r,

$$\text{X} = 4\pi g \int \frac{rdr}{(g^2+r^2)^{\frac{3}{2}}} = 4\pi g \left\{\frac{1}{g} - \frac{1}{\sqrt{g^2+a^2}}\right\}$$

$$= 4\pi \left\{1 - \frac{1}{\sqrt{1+\frac{a^2}{g^2}}}\right\}.$$

Put $g = 0$; then, since a is finite, $\frac{a}{g} = \infty$, and therefore $\text{X} = 4\pi$.

There is a difficulty at this point in Poisson's analysis. He says, that he proposes taking $+\infty$ and $-\infty$ as the limits of integration for $z-$ limits; of which, to say the least, the admissibility ought to be proved,—because it is not necessary, in order that the elements of the integral should be sensible, that ω' should be indefinitely nearly equal to ω, but only that it should either be equal or differ from it by some even multiple of π. And hence if $+\infty$ and $-\infty$ be the limits of z, it seems hard to assert, *à priori*, that no elements will be included in the integral but such as properly belong to it. But Poisson does not, in fact, take the above limits, for he transforms the integral by putting

$$z^2(1-\mu^2) = \left(g^2 + \frac{\nu^2}{1-\mu^2}\right)x^2;$$

and then takes $\pm\infty$ as the limits of x. Now it will be observed, that x is multiplied by an evanescent quantity, and therefore it does not follow that z should be infinite because x is so.

The truth of these remarks will, I think, appear from observing the method above given of finding the value of X. We find that

$$X = 4\pi\left\{1 - \frac{1}{\sqrt{1+\frac{a^2}{g^2}}}\right\}.$$

The assumption of $\pm\infty$ for the limits of z, will be somewhat similar in its nature to that of 0 and ∞ for those of r; this will be effected by putting $a=\infty$, in which case we shall, it is true, get the same value of X as before; but the assumption is not necessary, since if a be finite, $\frac{a}{g}$ will be infinite, which is all we require. Moreover, if $a=\infty$, we should have $X=4\pi$, independently of the value of g; whereas the evanescence of g is an essential condition of the problem.

II. The case in which μ has one of the values $+1$ or -1, or ω one of the values 2π or 0, will require a few words.

1. Suppose $\omega=0$, then referring to the figure there will be *two* points Q, Q′, for which $\omega'=0$, and $\omega'=2\pi$, for which the elements of the integral will not vanish.

The limits of r may be the same as before, but those of θ must be $\pm\frac{\pi}{2}$; since otherwise we should integrate out of the rectangle *acbd*. The value of the integral for each point will be 2π, and therefore on the whole $X=4\pi$, as before.

The same remarks will apply when $\omega=2\pi$.

2. Suppose $\mu=1$. Then there will be one point, as R, (see fig. 2) on the circumference of *acdb*, for which the value of the elements will be sensible. It would appear at first sight that the limits of θ should be 0 and π in this case, and thus we should have $X=2\pi$; that this is not the case, may be easily shewn by an independent investigation, as in Pratt (page 168), and the reason of this may perhaps be given thus: we have assumed

$$\nu' = \frac{\nu}{\sqrt{1-\mu^2}},$$

$$\text{and} \quad \nu' = r\sin\theta;$$

$$\therefore \nu = \sqrt{1-\mu^2}\,.\,r\sin\theta.$$

In the particular case, therefore, of $\mu=1$, we see that $\nu=0$ whatever be θ; and we shall therefore not go beyond our limits (or integrate outside the rectangle *acdb*), even though we take 0 and 2π as the limits of θ.

The same explanation will apply when $\mu=-1$.

H. G.

VII.—ON THE DEVELOPMENT OF THE SQUARE ROOTS OF INTEGRAL AND FRACTIONAL NUMBERS BY CONTINUED FRACTIONS.

By JAMES BOOTH, M.A., Principal of and Professor of Mathematics in Bristol College.

THAT the square root of a number may be exhibited in the form of a continued fraction, is a property of such fractions long known, yet the proof usually given, showing that the terms of such fraction are periodic, appears both tedious and obscure; and the inverse problem, to determine the numbers whose square roots may be developed in periods of one, two, three, or more given terms, appears never to have attracted the attention of mathematicians: to discuss this is the object of the present paper.

Assuming as known the common properties of continued fractions, let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$, be the three final consecutive converging fractions of the period, and let μ be the last quote of the period;

$$\text{then} \quad R=Q\mu+P, \quad R'=Q'\mu+P' \ldots\ldots \ldots(1),$$

α, β, γ, $\delta \ldots \mu$ being the successive quotes of the period.

The rule by which these converging fractions are found is—multiply the numerator of the converging fraction just found by the corresponding quote, and add the numerator of the preceding fraction; this sum will be the numerator of the next converging fraction: and the denominator may be found in a precisely similar manner.

Let N be the number whose square root is required, a^2 a square contained in N, which, when the number is an integer, will be the greatest possible, k the difference between N and a^2, so that

$$N = a^2 + k \ldots\ldots\ldots\ldots (2),$$

$$\text{and let} \quad \sqrt{N} = a + z \ldots\ldots\ldots\ldots (3),$$

$$\text{then} \quad z = \cfrac{1}{\alpha + \cfrac{1}{\beta + \ldots \atop \ldots\ldots \ldots \cfrac{+1}{\mu + z}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now the final converging fraction of the first period is

$$\frac{R}{R'} = \frac{Q\mu+P}{Q'\mu+P'},$$

which involves μ only in the first power; hence, as μ and z are

similarly involved in (4), the complete value of the final converging fraction is

$$\frac{Q(\mu+z)+P}{Q'(\mu+z)+P'};$$

equating this expression with z, we find

$$z=-\left(\frac{R'-Q}{2Q'}\right)+\sqrt{\left(\frac{R'-Q}{2Q'}\right)^2+\frac{R}{Q'}} \quad \ldots . \ (5),$$

but from (3) $\qquad z=-a+\sqrt{N}.$

Equating those values of z, comparing the rational and irrational parts together, we find

$$a=\frac{R'-Q}{2Q'} \quad \ldots \ldots \ldots \ldots (6),$$

$$N=\left(\frac{R'-Q}{2Q'}\right)^2+\frac{R}{Q'} \ldots \ldots \ldots \ldots (7);$$

or, in the last equation, introducing the value of $\left(\frac{R'-Q}{2Q}\right)$ given by (6), we get

$$N=a^2+\frac{R}{Q'} \ldots \ldots \ldots \ldots (8);$$

but by (2),

$$N=a^2+k, \quad \text{or } k=\frac{R}{Q'} \ldots \ldots \ldots (9):$$

hence we obtain the following Theorem:

In reducing the square root of a number, whether integral or fractional, to the form of a continued fraction, the numerator of the final converging fraction of a period bears to the denominator of the preceding converging fraction a given ratio.

From equation (6) we find $R'=2aQ'+Q$; but by the general rule for the formation of the converging fractions, $R'=\mu Q'+P'$; hence, eliminating R' we find

$$a=\frac{\mu}{2}+\frac{P'-Q}{2Q'} \ldots \ldots (10).$$

Now when N is an integer a must be so also; and that this may be always the case, P' must be equal to Q: whence it follows that $\mu=2a$, or *when the square root of any integer is developed in the form of a continued fraction, the final quote μ of the period is always double the integral part of the root, and the numerator of the last converging fraction of the period but one is equal to the denominator of the last converging fraction but two.* Hence μ is always an even number, let it be $=2\lambda$.

To find the form of those integral numbers whose square roots may be developed in periods consisting of only one term,

let the quotes be $0,\ 2\lambda,$

and the converging fractions $\frac{1}{0},\ \frac{0}{1},\ \frac{1}{2\lambda};$

hence $R'=2\lambda,\ R=1,\ Q'=1,\ Q=0,$

or $\frac{R'-Q}{2Q'}=\lambda$ an integer, and $k=\frac{R}{Q'}=1$; therefore

$$N=(\lambda^2+1);$$

or all numbers of the form (λ^2+1) may have their square roots developed under the form of a continued fraction, the period consisting of only one term; thus

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+1\ \&c.}} \qquad \sqrt{17}=4+\cfrac{1}{8+\cfrac{1}{8+1,\ \&c.}}$$

To find the form of those integral numbers whose square roots may be developed in periods of two terms,

let the quotes be $0,\ a,\ 2\lambda,$

and the corresponding converging fractions $\frac{1}{0},\ \frac{0}{1},\ \frac{1}{a},\ \frac{2\lambda}{2\lambda a+1};$

or $R'=2a\lambda+1,\ R=2\lambda,\ Q'=a,\ Q=1;$

hence $a=\frac{R'-Q}{2Q'}=\lambda$, and $k=\frac{R}{Q'}=\frac{2\lambda}{a}$:

hence 2λ must be a multiple of a.

Let $\lambda=ta$; then all numbers of the form (t^2a^2+2t), where a and t are any integral numbers whatever, may have their square roots developed in periods of two terms. Thus, let $a=5,\ t=1,$

$$\sqrt{27}=5+\cfrac{1}{5+\cfrac{1}{10+\cfrac{1}{5}}}$$

let $a=3,\ t=2,$

$$\sqrt{40}=6+\cfrac{1}{3+\cfrac{1}{12+\cfrac{1}{3}}}$$

When the period is to consist of three terms, to find the form of the numbers,

let the quotes be $0,\ a,\ \beta,\ 2\lambda,$

and the converging fractions $\frac{1}{0},\ \frac{0}{1},\ \frac{1}{a},\ \frac{\beta}{a\beta+1},\ \frac{2\beta\lambda+1}{2a\beta\lambda+2\lambda+a}:$

then, as the number N is an integer, $Q=P'$, or $a=\beta$,

and $\frac{R}{Q}=k=\frac{2a\lambda+1}{a^2+1}$, putting a for β;

hence $\dfrac{2a\lambda+1}{a^2+1}$ must be an integer. Write this fraction in the form

$$\frac{2a\lambda+a^2+1-a^2}{a^2+1}=k, \quad \text{or} \quad \frac{a(2\lambda-a)}{a^2+1}=k-1\,;$$

hence $(2\lambda-a)$ must be a multiple of (a^2+1), or $2\lambda-a=t(a^2+1)$, or $\lambda=\dfrac{ta^2+t+a}{2}$; hence both t and a must be even.

$$\text{Let } t=2\tau \text{ and } a=2\delta,$$

$$\text{then } \lambda=4\tau\delta^2+\tau+\delta \text{ and } k=4\tau\delta+1\,;$$

hence $\quad N=(\lambda^2+k)=(\delta^2+1)+2\delta(4\delta^2+3)\tau+(4\delta^2+1)^2\tau^2.$

Thus, let $\delta=1$, $\tau=1$,

$$\sqrt{41}=6+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{12}+\&c.}}$$

let $\delta=2$, $\tau=2$,

$$\sqrt{1313}=36+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{72}}}$$

When the period consists of four terms, let the quotes and corresponding converging fractions be

$$\begin{array}{cccccc} 0, & a, & \beta, & \gamma & 2\lambda & \\ \dfrac{1}{0}, & \dfrac{0}{1}, & \dfrac{1}{a}, & \dfrac{\beta}{a\beta+1}, & \dfrac{\beta\gamma+1}{a\beta\gamma+a+\gamma}, & \dfrac{2\beta\lambda\gamma+2\lambda+\beta}{2a\beta\gamma\lambda+2a\lambda+2\gamma\lambda+a\beta+1}. \end{array}$$

Now as the required number is to be integral,

$$Q=P', \text{ or } \beta\gamma+1=a\beta+1, \text{ or } \gamma=a\,;$$

hence the third term of the period must be equal to the first: and as $\dfrac{R}{Q'}$ must be an integer $\dfrac{2\lambda a\beta+2\lambda+\beta}{a^2\beta+2a}$ must be an integer k, putting a for γ.

$$\text{Let } \lambda=\nu a, \quad \text{then } k=2\nu-\frac{2\nu a-\beta}{a^2\beta+2a};$$

$$\text{and as } \frac{2\nu a-\beta}{a^2\beta+2a} \text{ must be an integer, let } \beta=\delta a\,;$$

$$\text{then } \frac{2\nu-\delta}{\delta a^2+2} \text{ must be an integer } =\eta\,;$$

$$\text{hence} \quad 2\nu=\delta+2\eta+\eta\delta a^2\,;$$

$$\text{hence} \quad k=\delta+\eta+\delta\eta a^2, \quad \text{and } \lambda=\eta a+\delta a\left(\frac{1+\eta a^2}{2}\right),$$

therefore $N=(\lambda^2+k)=\eta\,(1+\eta a^2)+(4\delta+\delta^2 a^2)\left(\frac{1+\eta a^2}{2}\right)^2$.

Let $a=2$, $\delta=2$, $\eta=1$, then $N=155$; therefore

$$\sqrt{155}=12+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{24}}}}$$

To find the number whose square root may give the quotes 1, 2, 3, 4; the quotes and converging fractions are

$$0,\quad 1,\quad 2,\quad 3,\quad 4,$$
$$\frac{1}{0},\ \frac{0}{1},\ \frac{1}{1},\ \frac{2}{3},\ \frac{7}{10},\ \frac{30}{43}.$$

Now the integral part of the root is $\frac{R'-Q}{2Q'}=\frac{43-7}{2\,.\,10}=\frac{9}{5}$,

and $k=\frac{R}{Q'}=3$; hence $N=a^2+k=\frac{156}{25}$: consequently

$$\sqrt{\frac{156}{25}}=\frac{9}{5}+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{1+\&c.}}}}}$$

The square root of any number being developed in the form of a continued fraction, such fraction will be periodic.

Assume the equations (6), (8),

$$\left.\begin{array}{l} N=a^2+\frac{R}{Q'},\quad R'=2aQ'+Q, \\ \text{and } R'Q-RQ'=\pm 1, \end{array}\right\}\ \ldots\ldots\ldots(A);$$

the upper sign to be taken when $\frac{R}{R'}$ holds an *even* place in the series of fractions $\frac{1}{0}$, $\frac{0}{1}$, ... $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$, and *vice versâ*. From these three equations, eliminating R and R′, we find

$$Q=\sqrt{NQ'^2\pm 1}-aQ'\ \ldots\ldots(11).$$

Now such a value for Q' must be found as will render $(NQ'^2\pm 1)$ a complete square, and greater than $a^2Q'^2$; and as such a value of Q' is always possible, it is manifest that neither the numerator nor denominator of the last converging fraction of the period can ever become infinite: thence, as the numerators and denominators of

these converging fractions, from the law of their formation, are continually increasing in magnitude, it is evident that a finite number of converging fractions must intervene between the first and last of the period; and as a partial quote exists for every such fraction, it follows that the number of such quotes is finite, or the period is finite.

Hence we may derive a singular theorem, namely, that the numbers which, substituted for t, will render the formula $(At^2 \pm 1)$ a perfect square, are the denominators of the penultimate converging fractions in each period of the development of the square root of A.

Thus, let $A=11$; then the converging fractions are

$$\frac{1}{3}, \frac{6}{19}, \frac{19}{60}, \frac{120}{379}, \frac{379}{1197}, \text{\&c.}$$

it will be found that 3, 60, 1197, substituted for t, will render $(11t^2+1)$ a complete square.

By the aid of these principles the converging fractions of a period of the development of the square root of A may be found in an inverse order: thus, let $A=27$, then $\sqrt{(27Q'^2+1)}$ is rational when $Q'=5$, and

$$\sqrt{27 \,.\, Q'^2 + 1} = \sqrt{676} = 26,$$

$$Q = \sqrt{NQ'^2+1} - aQ' = 26 - 25 = 1,$$

$$\text{and } P' = 1, \quad \text{and } R = kQ' = 10, \quad \text{also } R' = 51;$$

hence the fractions are $\frac{10}{51}, \frac{1}{5}, \frac{1}{1}$.

Had Q and Q′ been eliminated from equations (A), we should have found

$$R = \sqrt{NR'^2 \mp k} - aR' \quad \text{.........} \quad (12),$$

$$\text{or} \quad R = \sqrt{NQ^2 \mp k} + aQ \quad \text{...........} \quad (13),$$

had we eliminated R′ and Q′.

Hence it follows, that if the square root of a number N be developed in the form of a continued fraction, the denominators of the last converging fractions of each period, or the numerators of the immediately preceding fractions, substituted for t, will render the formula $(Nt^2 \mp k)$ a complete square, k being the difference between N and the greatest square contained in it.

Thus, let $N=7$; here $k=3$, and the converging fractions being

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{9}{14} \Bigg| \frac{11}{17}, \frac{20}{31}, \frac{31}{48}, \frac{144}{223} \Bigg| \frac{175}{271}, \frac{319}{494}, \frac{494}{765}, \frac{2295}{3554} \Bigg| \text{\&c.}$$

it will be found that 2, 31, 494, or 14, 223, 3554, put for t in $(7t^2-3)$, will render this expression a square; or 3, 48, 765, put for t in $(7t^2+1)$, will constitute this formula a complete square.

VIII.—EXAMPLES OF THE DIALYTIC METHOD OF ELIMINATION AS APPLIED TO TERNARY SYSTEMS OF EQUATIONS.

By J. J. Sylvester, F.R.S., Professor of Natural Philosophy in University College, London.

This method is of universal application, and at once enables us to reduce any case of elimination to the form of a problem, where that operation is to be effected between quantities linearly involved in the equations which contain them.

As applied to a binary system, $fx = 0$, $\phi x = 0$, the method furnishes a rule by which we may unfailingly arrive at *the determinant*, free from every species of irrelevancy, whether of a linear, factorial, or numerical kind.

The rule itself is given in the *Philosophical Magazine*, (London and Edinburgh, Dec 1840). The principle of the rule will be found correctly stated by Professor Richelot, of Konigsberg, in a late number of *Crelle's Journal*, at the commencement of a memoir in Latin bordering on the same subject, ("Nota ad Eliminationem pertinens.")

My object at present is to supply a few instances of its application to ternary systems of equations.

Ex. 1. To eliminate x, y, z, between the three homogeneous equations.

$$\left.\begin{aligned} Ay^2 - 2C'xy + Bx^2 &= 0 \ldots\ldots (1), \\ Bz^2 - 2A'yz + Cy^2 &= 0 \ldots\ldots (2), \\ Cx^2 - 2B'zx + Az^2 &= 0 \ldots\ldots (3). \end{aligned}\right\}$$

Multiply the equations in order by $-z^2$, x^2, y^2, add together, and divide out by $2xy$; we obtain

$$C'z^2 + Cxy - A'xz - B'yz = 0 \ldots\ldots (4)$$

By similar processes we obtain

$$A'x^2 + Ayz - B'yx - C'zx = 0 \ldots\ldots (5),$$
$$B'y^2 + Bzx - C'zy - A'xy = 0 \ldots\ldots (6).$$

Between these (6), treated as simple equations, the six functions of x, y, z, viz. $x^2, y^2, z^2, xy, xz, yz$, treated as *independent* of each other, may be eliminated; the results may be seen, by mere inspection, to come out

$$ABC\,(ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C') = 0,$$

or rejecting the special (*n. b.* not *irrelevant*) factor ABC, we obtain

$$ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C' = 0.$$

I may remark, that the equations (1), (2), (3), or (4), (5), (6), express the condition of

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy,$$

having a factor $\lambda x + \mu y + \nu z$; a general symbolical formula of which I am in possession for determining in general the condition of any polynomial of any degree having a factor, furnishes me at once with either of the two systems indifferently. The aversion I felt to reject *either*, led me to employ both, and thus was the occasion of the Dialytic Principle of Solution manifesting itself.

Ex. 2.
$$Ax^2 + ayz + bzx + cxy = 0 \ldots\ldots\ldots (1),$$
$$My^2 + lyz + mzx + nxy = 0 \ldots\ldots\ldots (2),$$
$$Rz^2 + pyz + qzx + rxy = 0 \ldots\ldots\ldots (3).$$

Multiply equation (1) by $\beta y + \gamma z$, equations (2) and (3) by νz and κy respectively, and add the products together, we obtain terms of which y^2z and yz^2 are the only two into which x does not enter.

Make now the coefficients of each of these zero, and we have

$$a\gamma + l\nu + R\kappa = 0,$$
$$a\beta + M\nu + p\kappa = 0.$$

Let $\nu = a$, $\kappa = a$, then $\gamma = -(l + R)$, $\beta = -(M + p)$.
Hence, multiplying as directed, and then dividing out by x, we obtain

$$(m\nu + b\gamma)z^2 + (r\kappa + c\beta)y^2 + (b\beta + c\gamma + n\nu + q\kappa)yz$$
$$+ A\beta xy + A\gamma xz = 0,$$

or by substitution,

$$\{ra - c(M + p)\}y^2 + \{ma - b(l + R)\}z^2$$
$$+ \{an + aq - b(M + p) - c(l + R)\}yz$$
$$-A(M + p)xy - A(M + p)xz = 0 \ldots\ldots\ldots (4).$$

Similarly, by preparing the equations so as to admit in turns of y and z as a divisor, we obtain

$$\{ma - l(R + b)\}z^2 + \{mr - n(A + q)\}x^2$$
$$+ \{mc + mp - n(R + b) - l(A + y)\}xz$$
$$-M(R + b)yz - A(A + q)xy = 0 \ldots\ldots\ldots (5),$$
$$\{rm - q(A + n)\}x^2 + \{ra - p(M + c)\}y^2$$
$$+ \{rl + rb - p(A + n) - q(M + c)\}xy$$
$$-R(A + n)xz - R(M + c)yz = 0 \ldots\ldots\ldots (6).$$

Between the six equations (1), (2), (3), (4), (5), (6), x^2, y^2, z^2, xy, xz, yz, may be eliminated; the result will be a function of nine letters {three out of each equation (1), (2), (3),} equated to zero. *Perhaps* the determinant may be found to contain a special factor of three letters; and if so, may be replaced by a simpler function of six letters only.

Ex. 3. To eliminate between the three general equations

$$Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy=0,$$
$$Lx^2+My^2+Nz^2+2Pyz+2Qzx+2Rxy=0,$$
$$fx+gy+hz=0.$$

By virtue of *one* of the two canons which limit the forms in which the letters can appear combined in the determinant of a general system of equations, we know that the determinant in this case (freed of irrelevant factors) ought to be made up in every term of eight letters (powers being counted as repetitions), viz. (A, B, C, D, E, F,) must enter in binary combinations (L, M, N, P, Q, R,) the same, whereas f, g, h, must enter in *quaternary* combinations.

To obtain the determinant, write

$$Ax^2+By^2+Cz^2+Dyz+Ezx+Fxy=0\ldots\ldots(1),$$
$$Lx^2+My^2+Nz^2+Pyz+Qzx+Rxy=0\ldots\ldots(2),$$
$$fx^2+gxy+hzx \qquad =0\ldots\ldots(3),$$
$$fxy+gy^2+hzx \qquad =0\ldots\ldots(4),$$
$$fxz+gyz+hz^2 \qquad =0\ldots\ldots(5).$$

We want one equation more of *three* letters between x^2, y^2, z^2, xy, xz, yz. To obtain this, write

$$(Ax+Ez+Fy)\,x_{,}+(By+Fx+Dz)\,y_{,}+(Cz+Dy+Ex)\,z_{,}=0,$$
$$(Lx+Qx+Ry)\,x_{,}+(My+Rx+Pz)\,y_{,}+(Nz+Py+Qx)\,z_{,}=0,$$
$$fx_{,}+ \qquad gy_{,}+ \qquad hz_{,}=0,$$

Forget that $x_{,}=x$, $y_{,}=y$, $z_{,}=z$, and eliminate x, y, z, we obtain

$$h\begin{Bmatrix}(Ax+Ez+Fy)\,(My+Rx+Pz)\\-(By+Fx+Dz)\,(Lx+Qz+Ry)\end{Bmatrix}$$
$$+g\begin{Bmatrix}(Cz+Dy+Ex)\,(Lx+Qz+Ry)\\-(Nz+Py+Qx)\,(Ax+Ez+Fy)\end{Bmatrix}$$
$$+f\begin{Bmatrix}(Nz+Py+Qx)\,(By+Fx+Dz)\\-(Cz+Dy+Ex)\,(My+Rx+Pz)\end{Bmatrix}=0.$$

This may be put under the form

$$\alpha x^2+\beta y^2+\gamma z^2+\alpha' yz+\beta' zx+\gamma' xy=0\ \ldots\ldots\ldots(6),$$

where the coefficients are of the first order in respect to f, g, h, L, M, N, P, Q, R, A, B, C, D, E, F; in all of the third order.

Between the equations marked from (1) to (6), the process of linear elimination being gone through, we obtain as equated to zero a function of $5+3$, or of eight letters, two belonging to the first equation, two to the second, and four to the third; so that the determinant is clear of all factorial irrelevancy.

Ex. 4. To eliminate x, y, z between the three equations

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy=0,$$
$$Lx^2+My^2+Nz^2+2L'yz+2M'zx+2N'xy=0,$$
$$Px^2+Qy^2+Rz^2+2P'yz+2Q'zx+2R'xy=0.$$

Call these three equations $U=0$, $V=0$, $W=0$, respectively. Write

$$x.U=0\ldots\ldots(1),\quad y.U=0\ldots\ldots(2),\quad z.U=0\ldots\ldots(3),$$
$$x.V=0\ldots\ldots(4),\quad y.V=0\ldots\ldots(5),\quad z.V=0\ldots\ldots(6),$$
$$x.W=0\ldots\ldots(7),\quad y.W=0\ldots\ldots(8),\quad z.W=0\ldots\ldots(9).$$

We have here nine unilateral equations: one more is wanted to enable us to eliminate *linearly* the ten quantities

$$x^3,\ y^3,\ z^3,\ x^2.y,\ x^2.z,\ x.y^2,\ x.z^2,\ x.y.z,\ y^2.z,\ y.z^2.$$

This tenth may be found by eliminating $x, y, z,$ between the three equations

$$x(Ax+B'z+C'y)+y(By+C'x+A'z)+z(Cz+A'y+B'x)=0,$$
$$x(Lx+M'z+N'y)+y(My+N'x+L'z)+z(Nz+L'y+M'x)=0,$$
$$x(Px+Q'y+R'z)+y(Qy+R'x+P'z)+z(Rz+P'y+Q'x)=0;$$

for, by forgetting the relations between the bracketed and unbracketed letters, we obtain

$$(Ax+B'y+C'z)\left\{\begin{matrix}(My+N'x+L'z), & (Rz+P'y+Q'x),\\ -(Qy+R'x+P'z), & (Nz+L'y+M'x),\end{matrix}\right\}$$
$$+\ \&c.\ +\ \&c.\ =0,$$

which may be put under the form

$$ax^3+\beta y^3+\gamma z^3+\delta x^2y+\ldots\ldots=0^*\ldots\ldots\ldots\ldots(10).$$

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* We might dispense with a 10th equation, using the nine above given, to determine the ratios of the ten quantities involved to one another; and then by means of any such relations as

$$x^2y\times xy^3=x^2y^2\times x^2y^2,\ \text{ or }\ x^3\times y^3=x^2y\times xy^2,\ \&c.$$

obtain a determinant. But it is easy to see that this would be made up of terms, each containing literal combinations of the 18th order.

Again, we might use five out of the nine equations to obtain a new equation free from y^3, y^2z, yz^2, z^3; *i. e.* containing x in every term: which being divided by x, and multiplied by y, or by z, would furnish a 10th equation no longer linearly involved in the 9 already found. The determinant, however, found in this way, would consist of 14-ary combinations of letters.

Finally, we might, instead of a system of ten equations, employ a system of 15, obtained by multiplying each of the given three by any 5 out of the 6 quantities $x^2, y^2, z^2, xy, xz, yz$; but the determinant, besides being not *totally* symmetrical, would contain combinations of the 15th order.

I may take this opportunity of just adverting to the fact, that the method in the text does in fact contain a solution of the equation

$$\lambda U+\mu V+\nu W=x^r y^s z^t,$$

where $r+s+t=4$, and λ, μ, ν are functions of the second degree in regard to x, y, z to be determined.

By eliminating linearly between the equations marked from (1) to (10), we obtain as zero a quantity of the twelfth order in all, being of the fourth order in respect to the coefficients of each of the three equations, which is therefore the determinant in its simplest form.

I have purposely, in this brief paper, avoided discussing any theoretical question. I may take some other opportunity of enlarging upon several points which have hitherto been little considered in the theory of elimination, such as the Canons of Form, —the Doctrine of Special Factors,—the Method of Multipliers as extended to a system of any order,—the Connexion between the method of Multipliers and the Dialytic Process,—the Idea of Derivations and of Prime Derivatives extended to ultra-binary Systems. For the present I conclude with the expression of my best wishes for the continued success of this valuable Journal.

22, *Doughty-Street, London, January* 30, 1841.

IX.—ON THE EXISTENCE OF POSSIBLE ASYMPTOTES TO IMPOSSIBLE BRANCHES OF CURVES.

Let $f(x, y) = 0$ represent the equation to any curve. Let $Ax^{\alpha} + Bx^{\beta}$ be the two first terms in a development for y by descending powers of x. Then if $\alpha=1$, $\beta=0$, and A, B, be both possible quantities, the equation

$$y = Ax + B$$

corresponds to a rectilinear asymptote, within the plane of x, y, to an infinite branch of the curve. In case all the coefficients of the descending powers of x, after the two first, be also possible, then the branch lies wholly within the co-ordinate plane from some point at a finite distance from the origin to its ultimate coalescence with the asymptote. If, however, any of the following coefficients be impossible, then the branch will be wholly without the plane of co-ordinates between such limits. We propose to illustrate the case of the impossible coefficients, by the discussion of the equations to two appropriate curves.

Suppose (λ) (μ) (ν) to be particular values of λ, μ, ν, which satisfy the proposed equation, the general values are of the form

$$(\lambda) + AV - BW,$$
$$(\mu) + CW - AU,$$
$$(\nu) + BU - CV,$$

U, V, W mean the same as above: A, B, C are arbitrary constants. (λ) (μ) (ν) may easily be found by *analysing* the method applied to example (4).

Ex. 1. Take the equation

$$x^2y^2-2bx^2y-b^2y^2-2a^2xy+b^2x^2+2b^3y+2a^2bx+2a^4-b^4=0 \ldots(1);$$

arranging by powers of y, we have

$$(x^2-b^2)\,y^2-2\,(bx^2+a^2x-b^3)\,y+b^2x^2+2a^2bx+2a^4-b^4=0,$$

and therefore, after the execution of obvious simplifications,

$$(x^2-b^2)^2\,y^2-2\,(x^2-b^2)\,(bx^2+a^2x-b^3)\,y+(bx^2+a^2x-b^3)^2$$
$$=2a^4b^2-a^4x^2,$$

and therefore

$$(x^2-b^2)\,y=bx^2+a^2x-b^3\pm a^2\,(2b^2-x^2)^{\frac{1}{2}} \ldots\ldots.(2).$$

Suppose now that $x=\pm b$; then, since the coefficient of y becomes zero, and the right-hand member of the equation remains finite, it is clear that $y=\infty$. Hence the curve has two asymptotes, whose equations are

$$x=\pm b \;\ldots\ldots\ldots.(3).$$

Again, suppose that x is indefinitely great; then retaining only the highest power of x in the coefficient of y and in the right-hand member of the equation, we have

$$x^2y=bx^2,$$

and therefore $\qquad y=b \ldots\ldots\ldots\ldots (4),$

which is the equation to an asymptote.

It is evident, however, from the equation (2), that y has impossible values for all indefinitely large values of x. Hence the equation (4) belongs to the possible asymptote of an impossible branch of the curve.

In order to place in the clearest point of view the character of the impossible branch, we will suppose x to be susceptible of every degree of quantitive magnitude affected by the sign + or −, and transform accordingly the equation (1) from affectional to quantitive co-ordinates. The formulæ of transformation, (see a paper in the 9th number of this Journal, on the General Theory of the Interpretation of Equations in Algebraic Geometry,) are the following:—

$$\left.\begin{array}{lll} x=a, & y=\beta\,(\cos 2s\pi+\sqrt{-1}\,\sin 2s\pi), & \\ x'=a, & y'=\beta\cos 2s\pi, & z'=\beta\sin 2s\pi, \end{array}\right\}\ldots\ldots(5).$$

Substituting the expressions for x, y, in the equation (1) arranged by powers of y, and equating separately to zero the possible and the impossible parts of the result, we have

$$(a^2-b^2)\,\beta^2\cos 4s\pi-2\,(ba^2+a^2a-b^3)\,\beta\cos 2s\pi$$
$$+b^2a^2+2a^2ba+2a^4-b^4=0\ldots\ldots(6),$$

and $\qquad (a^2-b^2)\,\beta^2\sin 4s\pi-2\,(ba^2+a^2a-b^3)\,\beta\sin 2s\pi=0,$

or $\qquad \{(a^2-b^2)\,\beta\cos 2s\pi-(ba^2+a^2a-b^3)\}\sin 2s\pi=0\ldots(7).$

The equation (7) is satisfied by either of the two relations

$$\sin 2s\pi=0\;\ldots\ldots\ldots(8),$$

or $(a^2-b^2)\,\beta\cos 2s\pi-(ba^2+a^2a-b^3)=0\ldots\ldots(9).$

From (8) we have $2s\pi=\lambda\pi$, where λ is any integer whatever; hence from (5) and (6) we get

$$x'=a,\ \ y'=\beta\cos\lambda\pi,\ \ z'=0,$$

$$(a^2-b^2)\,\beta^2\cos 2\lambda\pi-2\,(ba^2+a^2a-b^3)\,\beta\cos\lambda\pi$$
$$+b^2a^2+2a^2ba+2a^4-b^4=0,$$

and therefore

$$\left.\begin{aligned}&(x'^2-b^2)\,y'^2-2\,(bx'^2+a^2x'-b^3)\,y'+b^2x'^2+2a^2bx'+2a^4-b^4=0\\ &\text{and}\quad z'=0\,;\end{aligned}\right\}\ldots\ldots\ldots(10),$$

again, from (5), (6), (9), we have

$$(x'^2-b^2)\,y'=bx'^2+a^2x'-b^3\ldots\ldots(11),$$

and

$$(x'^2-b^2)\,(y'^2-z'^2)-2\,(bx'^2+a^2x'-b^3)\,y'$$
$$+b^2x'^2+2a^2bx'+2a^4-b^4=0,$$

and therefore, by combining the two,

$$(x'^2-b^2)\,(y'^2+z'^2)=b^2x'^2+2a^2bx'+2a^4-b^4\,;$$

whence by (11)

$$(x'^2-b^2)^2z'^2+(bx'^2+a^2x'-b^3)^2=(x'^2-b^2)(b^2x'^2+2a^2bx'+2a^4-b^4),$$

and therefore

$$(x'^2-b^2)^2\,z'^2=a^4\,(x'^2-2b^2)\ldots\ldots\ldots(12).$$

The equations (10) represent the branches SK*s*, S′K′*s*′, in the plane of x', y', (See Fig. 3.) The straight lines SS′, *ss*′, which belong to the equations (3), are asymptotes to these branches.

The equations (11), (12), represent the branches TLK*l*, T′L′K′*l*′, of which KlT, K′l′T′, are the projections on the plane of x', y'. The straight line TT′ corresponding to the equation (4), is an asymptote to both these branches.

The equation (1) may be seen in Cramer's *Analyse des Lignes Courbes*, where, having defined the locus of (4) as being an equation of the form which corresponds to a rectilinear asymptote, he observes that, since the values of y in (1) for indefinitely great values of x, are impossible, "cette prétendue asymptote TT′ n' est accompagnée d' aucune branche infinie de la courbe."

Ex. 2. Take as another instance the equation

$$x^4\,(y-bx-c)^2=a^2-x^2\ldots\ldots\ldots(1).$$

Writing this equation under the form

$$(y-bx-c)^2=\frac{a^2-x^2}{x^4}\ \ldots\ldots\ldots(2),$$

it is clear that when x becomes indefinitely great

$$y=bx+c\ \ \ldots\ldots\ldots\ldots\ldots(3),$$

which is therefore the equation to an asymptote in the primary co-ordinate plane. But it is clear that when x is very great the values of y in (2) are impossible. Hence the equation (3) belongs to the possible rectilinear asymptote of an impossible branch.

If, as in the former example, we transform the equation (1), in which x is supposed to receive every degree of quantitive magnitude affected by + or −, from affectional to quantitive co-ordinates, we shall obtain the two following pairs of equations:

$$\left.\begin{array}{r} x'^4(y'-bx'-c)^2=a^2-x'^2, \\ z'=0, \end{array}\right\} \cdots\cdots\cdots (4),$$

$$\left.\begin{array}{r} \text{and} \quad y'=bx'+c, \\ x'^4z'^2=x'^2-a^2, \end{array}\right\} \cdots\cdots\cdots (5).$$

The equations (4) correspond to the branches BAb, B$'$A$'b'$, (Fig. 4,) in the plane of x', y', to which the axis of y' is asymptotic; and the equations (5) to the branches TKAk, T$'$K$'$A$'k'$, which have an asymptote TT$'$ in the plane of x', y', their projections upon this plane coinciding with TT$'$.

W. W.

X.—MATHEMATICAL NOTES.

1. A short mode of reducing the square root of a number to a continued fraction.

The common mode of proceeding by successive similar operations may be thus reduced to a rule, which will best be described by an instance. Let the number be 43,

$$\sqrt{43} = 6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5} + \&c.}}}}$$

6	1	5	4	5	5	4	5	1	6	6	1, &c.
1	7	6	3	9	2	9	3	6	7	1	7, &c.
6	1	1	3	1	5	1	3	1	1	12	1, &c.

Set down always in the first column the integers of the root, a unit, and the integers again. Form each column after the first from the preceding in the following manner:—

$$c, \quad c', \quad a' = \frac{43-c^2}{a}, \text{ which is always integer,}$$

$$a, \quad a', \quad b' = \text{integer of } \frac{6+c}{a'},$$

$$b, \quad b', \quad c' = a'b' - c.$$

The first figure of the third row is always the integer, and the last figure twice the integer; and the intermediate figures always shew the same series, whether reckoned from the beginning or the end. As soon as a succession of two similar numbers is seen in the first or second rows, it is a sign that the middle of the period is attained: if the reiteration take place in the first row, there will be an odd number of figures the same in the middle of the period; and if in the second row, an even number. In the last column but one of the whole period, $6+c$ will be divisible by a', but the converse is not true.

A. D. M.

2. *Irrationality of* $\varepsilon = 271828$ ... To the demonstration usually given, that ε is incommensurable, may be added this—that it cannot be the root of a quadratic equation, the coefficients of which are rational. If so it would satisfy the equation

$$a\varepsilon + \frac{b}{\varepsilon} = c,$$

a being a positive integer, and b and c integers either positive or negative. If then we replace ε by its equivalent series, and then multiply both sides by $1.2\ldots n$, we find

$$\frac{a}{n+1}\left(1+\frac{1}{n+2}+\&c\right) \pm \frac{b}{n+1}\left(1-\frac{1}{n+2}+\&c.\right) = \mu$$

μ being an integer. But we can always make the factor $\pm\dfrac{b}{n+1}$ positive by taking n even when b is negative, and *vice versâ*. If now we suppose n very large, the first side of the equation is a fraction which cannot be equal to the integer μ. Hence follows the proposition.—*Liouville's Journal, May* 1840.

ERRATUM.

Page 232, line 25 from the bottom, *for* Dec. 1840, *read* Feb. 1840

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